# 10.1 Rates of Change

Def] <u>Tangent line</u> for $f(x)$ at $x=a$ is a line through $(a, f(a))$ with slope equal to the slope of $f$ at $x=a$

Def] Instantaneous rate of change of $f$ at $x=a$ is the slope of the tangent line at $x=a$.

Notation] This slope is denoted as

$$f'(a) \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=a}$$

Recall] Instantaneous rate of change is equal to the average rate of change from $a$ to $b$ when $b$ tends to $a$. In other words:

$$f'(a) = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}.$$

If we let $b = a+h$, then this becomes

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note] $f'(a)$ is also called the <u>derivative</u> of $f$ at $a$.

Ex] The resting heart rate of mammals with respect to body mass can be modelled as

$R(x) = 200x^{-1/3}$, where $R(x)$ is in BPM and $x$ is in kg.

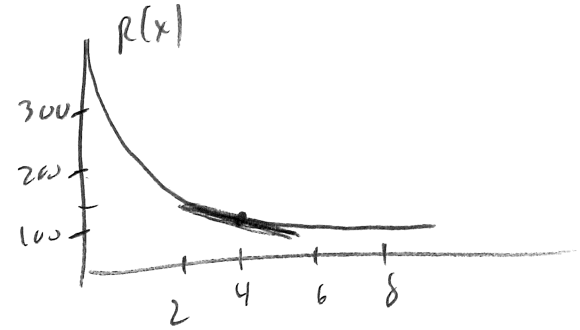
1) What are the units of $\frac{dR}{dx}$?

[Remark] If this were the average rate of change, then the units would be

$$\frac{\Delta y}{\Delta x} = \frac{BPM}{kg}.$$

units of $\frac{dR}{dx} = \frac{BPM}{kg}$

2) Graph the tangent line at $x=4$. Estimate the slope




Call the tangent line $T(x)$. We know $T(4) = R(4) = 126$, and can estimate $T(4.5)$ as 115.

So, $m = \frac{T(4.5) - T(4)}{4.5 - 4}$

$= \frac{115 - 126}{.5}$

$= -22 \frac{BPM}{kg}$

(approximately)

3) What does this mean?

For every 1 kg increase near $x=4$, the BPM decreases by 22 BPM.

4) Estimate $R(5)$ using the above info. Do not calculate $R(5)$ directly.

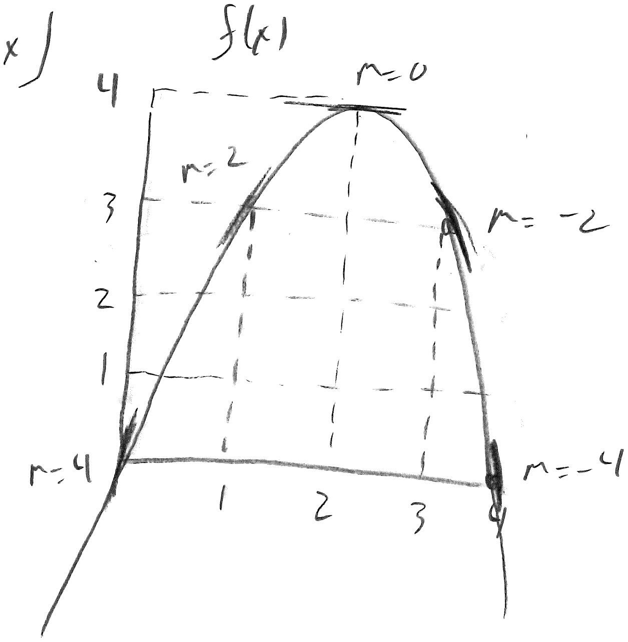
$R(4) = 126$; we change by $-22$ BPM if we increase by 1 kg. So,

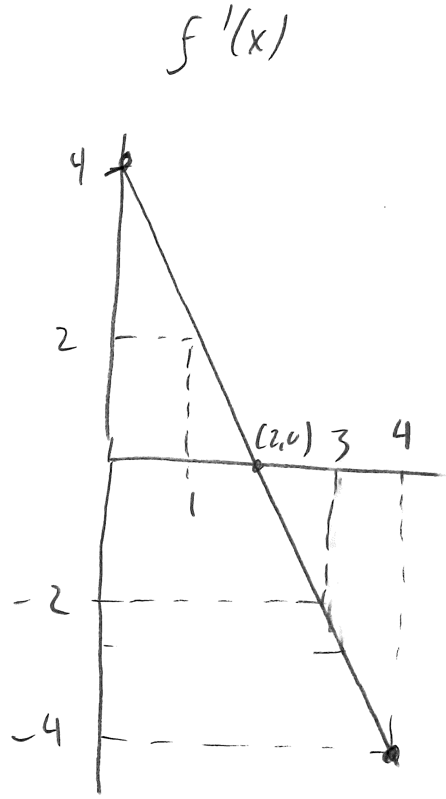
$$R(5) \approx 126 + (-22) = 104 \text{ BPM}$$

---

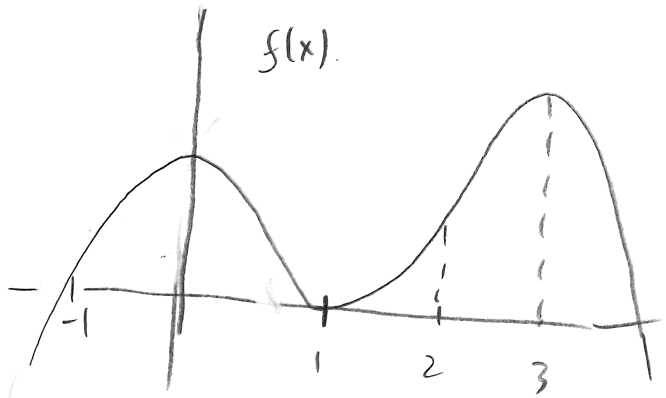
Graph of $f'(x)$.

Ex)



Ex] Graph $f'(x)$




How to sketch the derivative?

(1) Find where $f'(x)=0$

(2) Find where $f'(x)>0$ and $f'(x)<0$

(3) Find where $f(x)$ is steepest

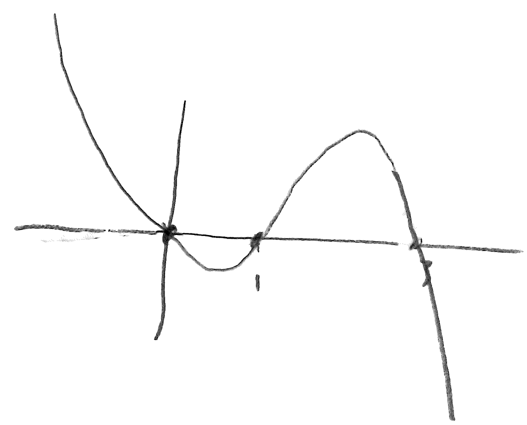
(4) Fill in the remaining details.

(1) $f'(x)=0$ at $x=0,1,3$

(2) $f'(x)>0$ on $(-\infty,0)$ $(1,3)$

$f'(x)<0$ on $(0,1)$, and $(3,\infty)$

(3) $f$ is steepest at either end




Remarks] $f(x)$ is increasing then $f'(x)>0$

$f(x)$ is decreasing then $f'(x)<0$

$f(x)$ has a horizontal tangent then $f'(x)=0$.

# Concavity

A function is concave up if $f$ is decreasing at a decreasing rate, or increasing at an increasing rate.

A function is concave down if $f$ is decreasing at an increasing rate, or increasing at a decreasing rate

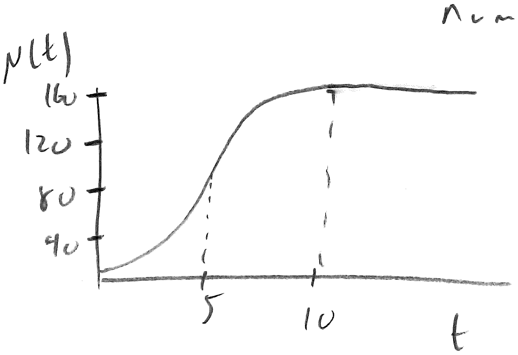
In other words,

$f$ is concave up if $f'$ is increasing

$f$ is concave down if $f'$ is decreasing.

$f$ switches concavity when $f'$ has a flat tangent line.

Ex] Deer population is plotted below. $N(t)$ is number of deer and $t$ is years




$N(t)$ is concave up on $(0,5)$ and concave down on $(5,10)$. It is flat otherwise.

In other words, the rate of growth of the deer population is increasing until $t=5$, and then starts to decrease.