4.1 Logarithms

Motivation: How to solve

$$10 = 2^t ?$$

If this was $10 = t^2$, then $t = \sqrt{10}$

(The inverse of $\cdot^2$ is $\sqrt{\cdot}$).

Let's introduce the inverse of exponentials.

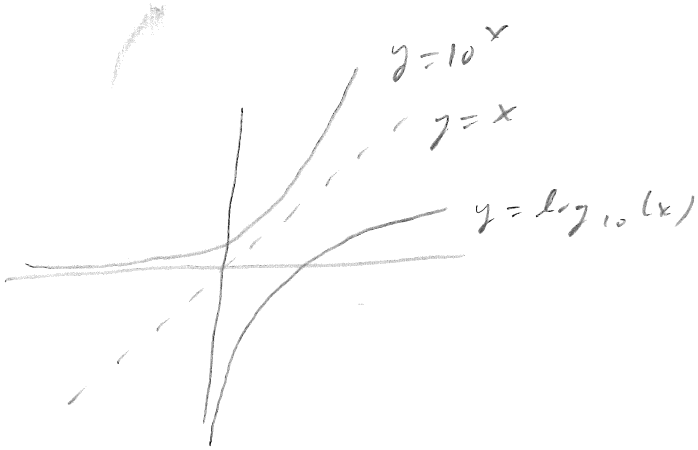
Def The base-b logarithm, $y = \log_b x$ is the inverse of $g(y) = b^y$.

In other words, $\log_b x = y$ if and only if $b^y = x$.

Ex $\log_{10} 100 = 2$ b/c $10^2 = 100$

$\log_b b = 1$ b/c $b^1 = b$.

Graphs



Note: logs are only defined for positive x.

End behavior: $x \to \infty$, $\log x \to \infty$

The graph $y = \log x$ always grows, but it grows really slowly.

Def Common logarithm: $y = \log x$ is defined as the base-10 logarithm (i.e. $\log x = \log_{10} x$)

natural logarithm: $y = \ln x$ is defined as the base-e logarithm (i.e. $\ln x = \log_e x$)

Ex $\ln e = 1$ (b/c $e^1 = e$)

$\log 10^t = t$ (b/c $10^t = 10^t$)

$2^{\log_2 t} = t$ (b/c $y = \log_2(t)$ is the power that 2 needs to be raised to equal t)

$\log_5 \frac{1}{25} = -2$ (b/c $5^{-2} = \frac{1}{25}$)

Problem: Find $3^{\log_3(5) + \log_3(7)}$

Answer: $3^{\log_3(5) + \log_3(7)} = 3^{\log_3(5)} \, 3^{\log_3(7)}$

$= 5 \cdot 7 = 35$

Domain of $\log_b(x) = y$

Remember, x must be positive. Why?

b/c $x = b^y > 0$ (recalling that $0 < b, b \neq 1$)

Ex] Find the domain of $\log_5(x-5)$.

The base doesn't matter here. Just need to solve $x - 5 > 0$, or, $x > 5$.

Change of base] What if we want to change the base b?

For instance, base-e $\longrightarrow$ base-10.

Recall that $e^{\ln t} = t$ and $10^{\log t} = t$.

So, $e^{(\ln 10)\log x} = \left(e^{\ln 10}\right)^{\log x}$

$= 10^{\log x}$

$= x.$

In other words, $(\ln 10)\log x$ is the power to which e is raised to equal x.

i.e. $(\ln 10)\log x = \ln x$ or

$$\log x = \frac{\ln x}{\ln 10}.$$

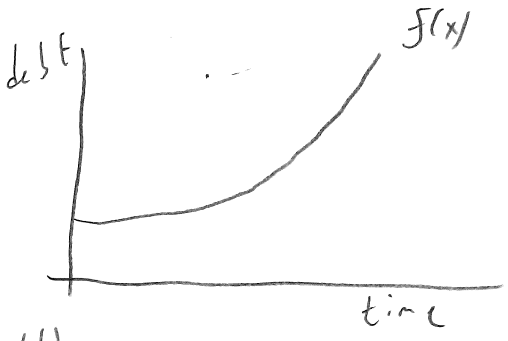
More generally, $\boxed{\log_b x = \dfrac{\log_a x}{\log_a b}}$

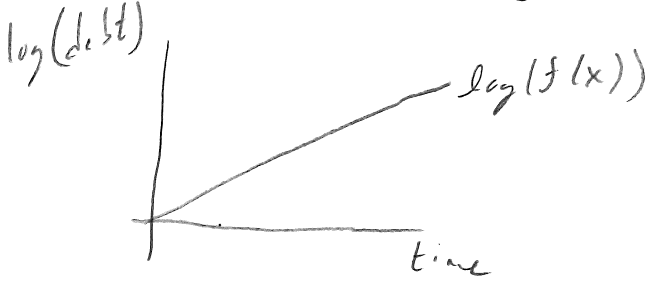
Debt example

$A_0 = 40,000$, $b = 1.07$ (i.e. 7% increase)

$x \sim$ years, $f(x) \sim$ debt owed

x	$f(x)$	$\log(f(x))$
0	40000	156.6
1	42800	157.6
2	45796	158.6
3	49002	159.6
10	78686	166.6
15	110361	171.6





log flattens exponentials

↑ (or linearizes)