5.5 Rational Functions

Def $R(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ are polynomials with no shared real zeros, is called a <u>rational function</u>.

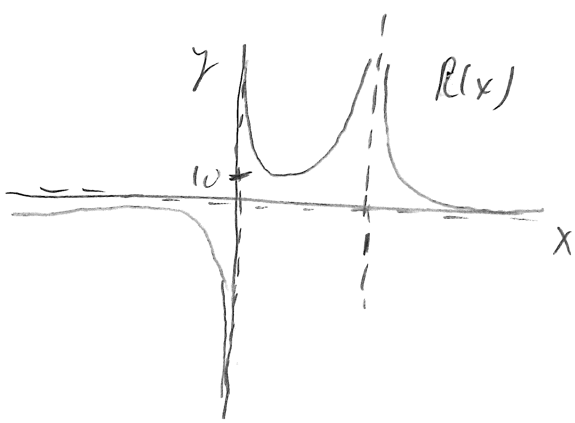
If c is such that $Q(c) = 0$, c is called a <u>pole</u>. The domain of $R(x)$ is all x excluding all possible c.

Ex $R(x) = \dfrac{x+1}{x^3 - 2x^2 + x}$

$$Q(x): x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

Poles are $x = 0$ and $x = 1$.

Domain $= (-\infty, 0) \cup (0, 1) \cup (1, \infty)$



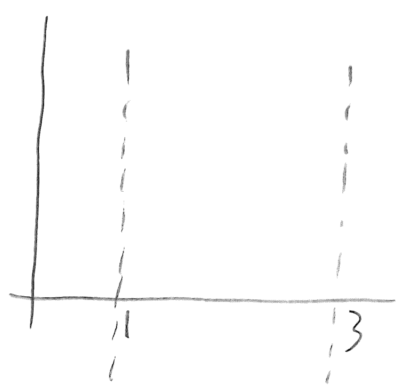
Def a vertical line that the graph approaches but never intersects is a <u>vertical asymptote</u>.

These occur at poles, and near a pole c,

$$R(c) \longrightarrow \pm\infty$$

Ex $R(x) = \dfrac{x-2}{(x-1)(x-3)}$

Poles at $x=1$ and $x=3$, $R(x)=0$ at $x=2$



What happens on either side of the lines?

Use a sign chart!

1 2 3

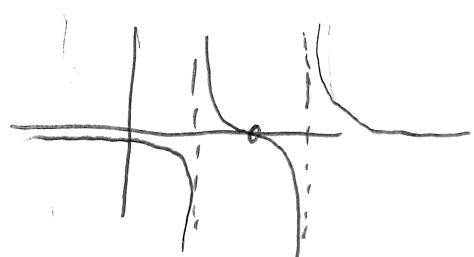
x	0	1.5	2.5	4
$R(x)$	$-$	$+$	$-$	$+$

Notice that $R(x)$ can only change sign at real zeros of numerator or denominator.

So, by the sign chart

$R(x) > 0$ on $(1,2) \cup (3,\infty)$

$R(x) < 0$ on $(-\infty,1) \cup (2,3)$



Q: How do we know what happens as $x \to \pm\infty$?

A: Take the limit of $R(x) = \frac{P(x)}{Q(x)}$.

Note: $\lim_{x \to \pm\infty} R(x) = \lim_{x \to \pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$

Ex) For $R(x) = \frac{x-2}{(x-1)(x-3)} = \frac{x-2}{x^2-4x+3}$

$$\lim_{x \to +\infty} R(x) = \lim_{x \to +\infty} \frac{x}{x^2} = 0$$

$$\lim_{x \to -\infty} R(x) = \lim_{x \to -\infty} \frac{x}{x^2} = 0.$$

Def) a horizontal line $y = b$ such that $R(x) \to b$ as $x \to \infty$ or $x \to -\infty$ (or both) is called a horizontal asymptote.

Note! We just care about the limit. $R(x)$ can cross HAs.

Ex] $f(x) = \dfrac{2x^2 - 8}{x^2 - 9}$

VA: ① poles: $x^2 - 9 = 0 \rightarrow x = \pm 3$

② VA at $x = 3, \; x = -3$

Zeros: $f(x) = 0 \rightarrow 2x^2 - 8 = 0 \rightarrow x = \pm 2$

Sign chart:

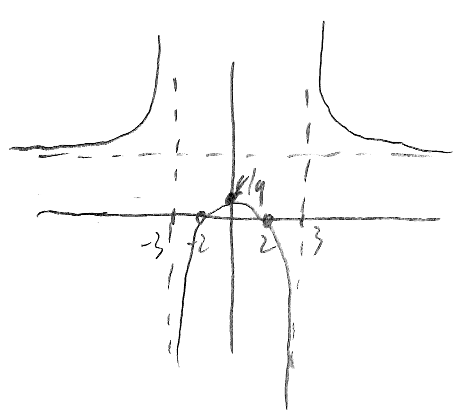
		-3		-2		2		3	
x	-4		-2.5		0		2.5		4
$f(x)$	+		-		+		-		+

HA: $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{2x^2 - 8}{x^2 - 9} = 2$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{2x^2 - 8}{x^2 - 9} = 2$

y-intercept: $f(0) = \dfrac{2(0)^2 - 8}{(0)^2 - 9} = \dfrac{8}{9}$

Graph:



Ex] $g(x) = \dfrac{2x^2 - 8x}{x^2 - 9}$

VA: (1) poles $x^2 - 9 = 0 \longrightarrow x = \pm 3$

(2) VA at $x = 3,\ x = -3$

Zeros: $g(x) = 0 \rightarrow 2x^2 - 8x = 0 \longrightarrow x(2x - 8) = 0 \longrightarrow x = 0,\ x = 4$

Sign chart

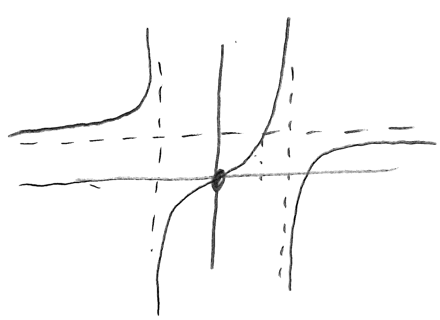
$-3 \quad 0 \quad 3 \quad 4$

x	-4	-2	2	3.5	5
$R(x)$	+	−	+	−	+

HA: $\lim\limits_{x \to \pm\infty} f(x) = 2$

y-intercept: $f(0) = \dfrac{2(0)^2 - 8(0)}{(0)^2 - 9} = 0$

Graph:



Find $g(x) = 2$: $2 = \dfrac{2x^2 - 8x}{x^2 - 9} \longrightarrow 2x^2 - 18 = 2x^2 - 8x$

$8x = 18$

$x = 9/4$