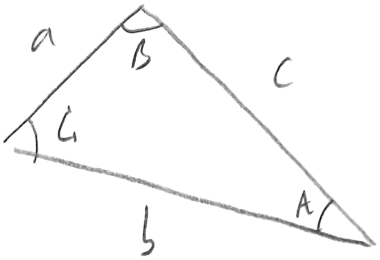
Chapter 9 Law of Cosines (9.1)

Law of Sines (9.2)



$c^2 = a^2 + b^2 - 2ab\cos C$ - Law of Cosines.

Notice that when $C = \frac{\pi}{2}$, this is just the pythagorean Theorem.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ - Law of Sines

Law of cosines:

Ex] Can be rewritten as

$$a^2 = b^2 + c^2 - 2bc\sin A$$

$$b^2 = a^2 + c^2 - 2ac\sin B$$

Ex] what if you wanted to solve for angle C?

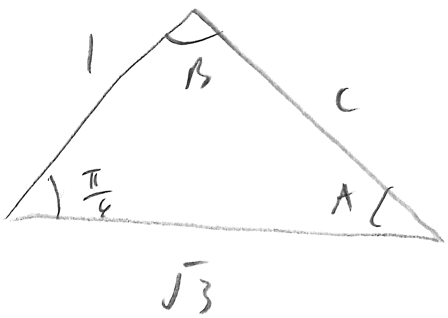
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\rightarrow \quad a^2 + b^2 - c^2 = 2ab\cos C$$

$$\rightarrow \quad \frac{a^2 + b^2 - c^2}{2ab} = \cos C$$

$$\rightarrow \quad \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = C$$

Ex] (SAS)



$a = 1$, $b = \sqrt{3}$, $c = ?$

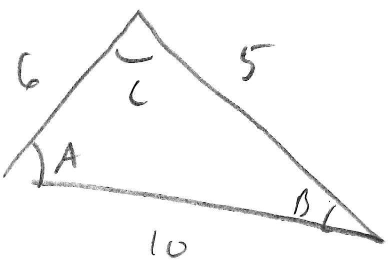
$A = ?$, $B = ?$, $C = \frac{\pi}{6}$

Solve for c.

$$c^2 = (1)^2 + (\sqrt{3})^2 - 2(1)(\sqrt{3}) \cos \frac{\pi}{6}$$

$$c^2 = 1 \rightarrow c = 1.$$

Ex] (SSS) Solve for A in degrees.



$a = 5$, $b = 6$, $c = 10$

$A = ?$, $B = ?$, $C = ?$

$$5^2 = 6^2 + 10^2 - 2(6)(10) \cos A$$

$$\rightarrow \cos(A) = \frac{111}{120}$$

$$\rightarrow A = \arccos\left(\frac{111}{120}\right)$$

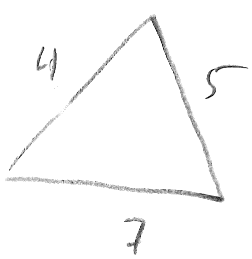
Ex] Heron's Formula

Def] $s = \frac{a+b+c}{2}$ is called the semiperimeter

The area of a triangle with semiperimeter s is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Example.



$$s = \frac{4+5+7}{2} = 8$$

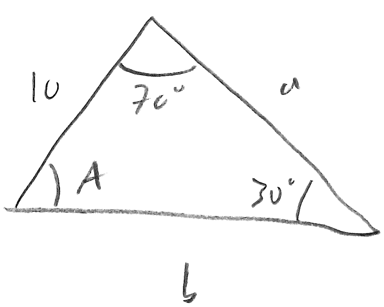
$$\text{Area} = \sqrt{8(8-4)(8-5)(8-7)} = \sqrt{96}.$$

Ex] Application of Law of Cosines

See Example 9.6 from the book.

Law of Sines

Ex] (SAA)



First, solve for A:

$$180° = 70° + 30° + A$$

$$A = 80°$$

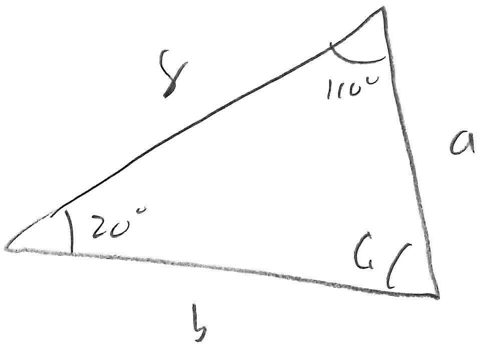
A = 80, B = 70
C = 30
a = ?, b = ?
c = 10

$$\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$b = \frac{c \sin B}{\sin C} = \frac{30 \sin 70}{\sin 30} = 18.8$$

$$a = \frac{c \sin A}{\sin C} = \frac{30 \sin 80}{\sin 30} = 19.7.$$

Ex] (ASA)



$a = ?, \ b = ?, \ c = 8$

$A = 20°, \ B = 110°, \ C = ?$

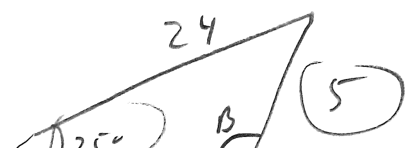
$180 = 20 + 110 + C \rightarrow C = 50°$

$$\frac{a}{\sin 20} = \frac{8}{\sin 50} \rightarrow a = 3.6$$

$$\frac{b}{\sin 110} = \frac{8}{\sin 50} \rightarrow b = 9.8$$

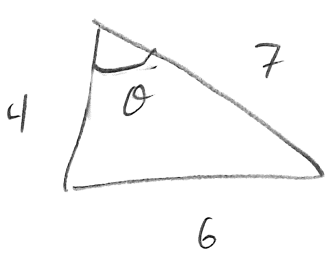
How to choose between Law of Cosines and Law of Sines

(1) angle-side pair



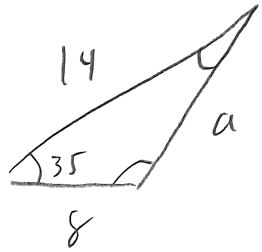
pair $\longrightarrow$ Law of sines to find B.

(2)



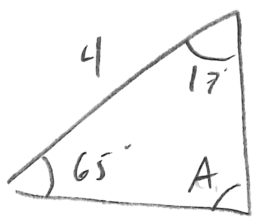
no pair $\longrightarrow$ Law of cosines to find θ.

(3)



no pair $\rightarrow$ use Law of Cosines

(4)



no pair, but can find A which makes a pair $\rightarrow$ Law of Sines

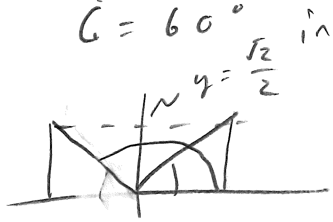
More Difficult examples

SSA triangles can have 0, 1, or 2 solutions.

Ex) Given angles A, B, and C, where $C = 30°$ and $\sin A = \frac{\sqrt{2}}{2}$, solve for A and B.

What if $C = 60°$ instead?

$\sin A = \frac{\sqrt{2}}{2}$



$A = 45°$ or $135°$

(1) $C = 30$: For $A = 45°$, $B = 180 - A - C = 105°$

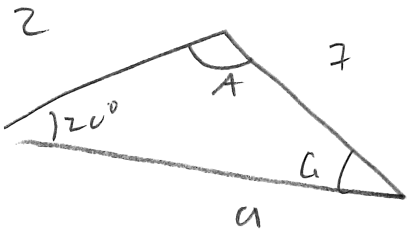
For $A = 135°$, $B = 180 - A - C = 15°$

(2) $C = 60°$: For $A = 45°$, $B = 180 - A - C = 75°$

For $A = 135°$, $B = 180 - A - C < 0$ $\leftarrow$ impossible to have.

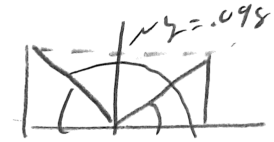
So only one solution.

Ex] Solve the triangle



$$\frac{\sin C}{2} = \frac{\sin(20°)}{7}$$

$$\sin C = \frac{2}{7}\sin(20°) = .098$$



$C_1 = \arcsin(.098) = 5.61°$

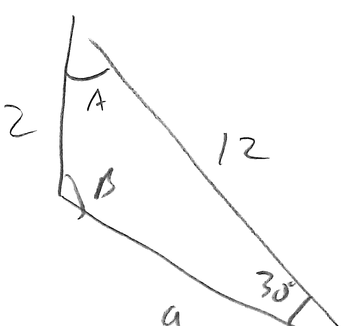
or $C_2 = 180 - 5.61° = 179.34.$

Since $C_2 + A + B > 180$, only C_1 is a valid solution.

So, $C = 5.61$; $B = 20°$; $A = 180 - B - C = 154.39°$

Find a: $\frac{\sin(20)}{7} = \frac{\sin(154.39)}{a} \rightarrow a = 8.85.$

Ex] Solve the triangle



$$\frac{\sin B}{12} = \frac{\sin(30°)}{2}$$

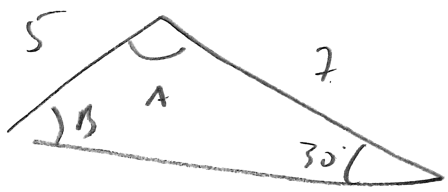
$$\downarrow$$

$$\sin(B) = 3$$

$-1 \leq \sin x \leq 1$, so this cannot happen.

No solution.

Ex/ Solve the triangle



$$\frac{\sin(B)}{7} = \frac{\sin(30)}{5}$$

$$\downarrow$$

$$\sin(B) = \frac{7}{10}$$

$$B_1 = \arcsin(\sin(B))$$
$$= 44.4°$$

$$A_1 = 180 - B_1 - C = 105.6°$$

$$B_2 = 180 - \arcsin(\sin(B)) = 135.6°$$

$$A_2 = 180 - B_2 - C = 14.4°$$

$$\frac{\sin 30}{5} = \frac{\sin 105.6}{a_1} \rightarrow a_1 = 9.6$$

$$\frac{\sin 30}{5} = \frac{\sin 14.43}{a_2} \rightarrow a_2 = 2.5$$