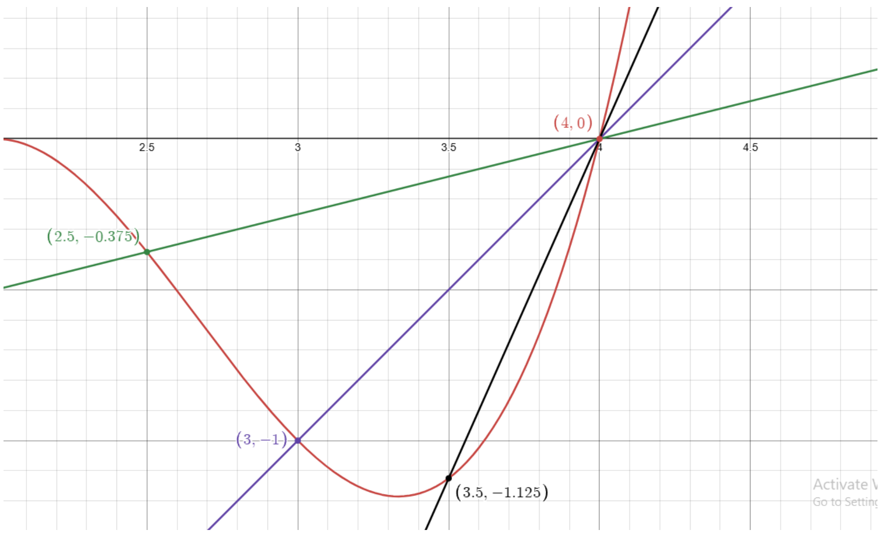
Section 1.3: Graphs and Rates of Change

Tangent Lines and Rates of Change

From Secant to Tangent Lines

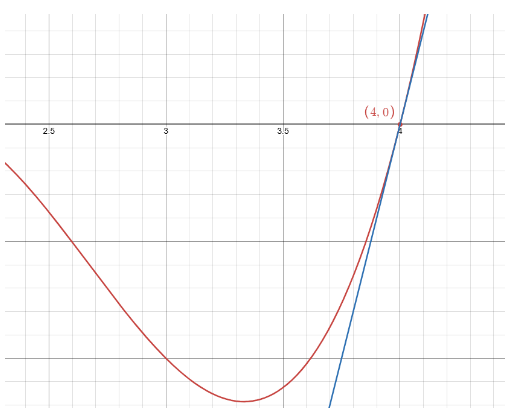
Question. What happens when we shrink the interval for the Average Rate of Change (AROC)?

Example.



As the interval around a point (x, y) on the curve of a function $f(x)$ shrinks, the secant lines passing through that point approach a single, unique line. This limiting line is called the **tangent line** to the curve at that point. The slope of the tangent line matches the slope of the function at that one point.

Example.



Definition 1. The slope of this tangent line is called the **instantaneous rate of change** of the function at that point.

Properties of Functions

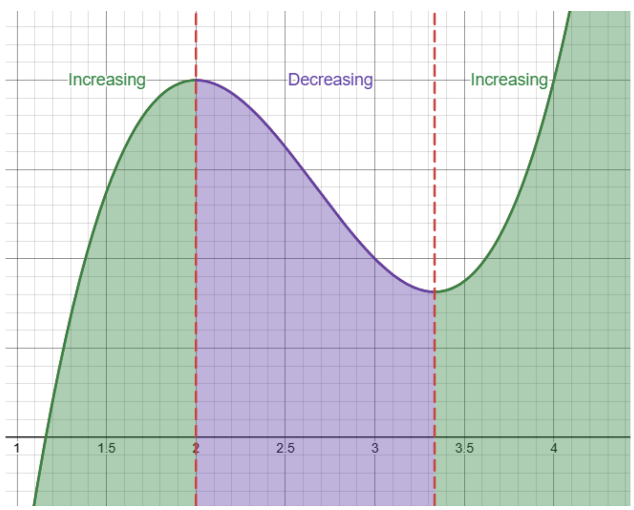
Question. What are Rates of Change used for?

Increasing and Decreasing Functions

Definition 2. An **increasing function** on an interval I is a function such that $f(x) < f(y)$ for any $x < y$ where both x, y are in I.

Definition 3. A **decreasing function** on an interval I is a function such that $f(x) > f(y)$ for any $x < y$ where both x, y are in I.

Example.



Observation:

- The slope of the tangent line (ROC) is **positive** for increasing functions.
- The slope of the tangent line (ROC) is **negative** for decreasing functions.

Question. What happens when the ROC changes from positive to negative (or vice-versa)?

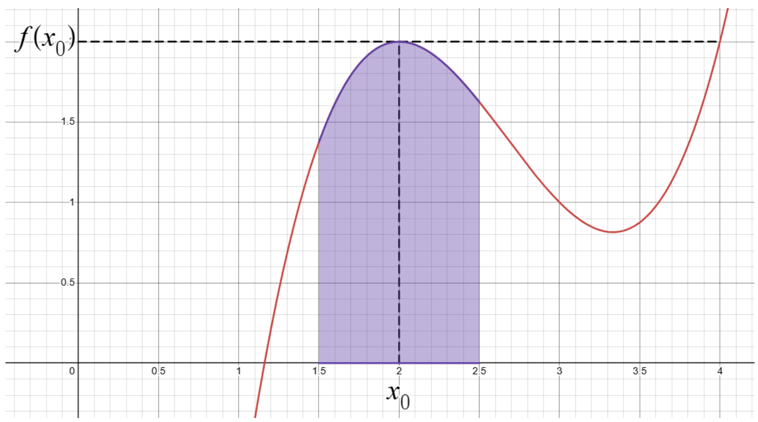
Local and Absolute Extrema

When the rate of change switches sign, the function may reach a local maximum or minimum.

Definition 4. A function f reaches a **local maximum** of $f(x_0)$ at $x = x_0$ if there is some interval around x_0 such that $f(x_0) \geq f(x)$ for any other x in that interval.

Definition 5. A function f reaches a **local minimum** at $x = x_0$ if there is some interval around x_0 such that $f(x_0) \leq f(x)$ for any other x in that interval.

Example.



In the interval $(1.5, 2.5)$, we see that $f(x_0)$ is the maximal value of f. So, x_0 is where f reaches a local maximum.

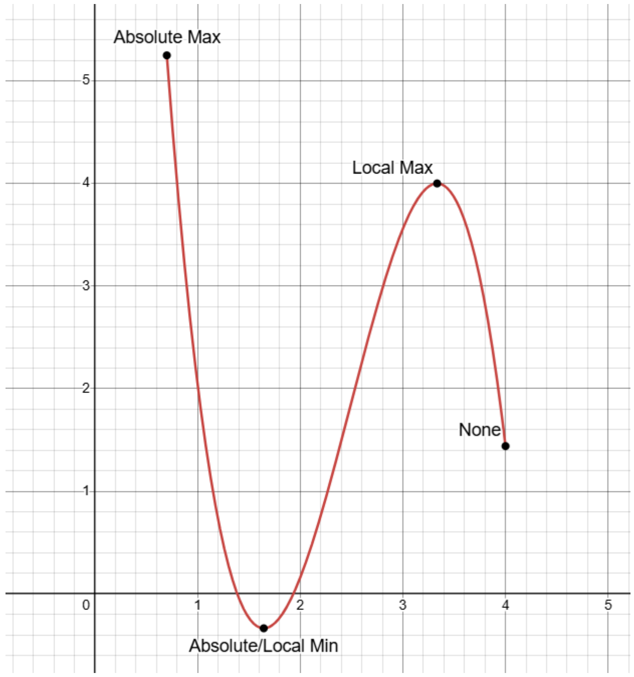
Why "local"? There may be other points outside this interval where the function's value is larger (for a maximum) or smaller (for a minimum).

Definition 6. A point x_0 is an **absolute maximum** if $f(x_0)$ is the highest value the function achieves on its entire graph.

Definition 7. A point x_0 is an **absolute minimum** if $f(x_0)$ is the lowest value the function achieves on its entire graph.

Note: Absolute minimums or maximums can occur at the endpoints of an interval.

Example.



Concavity

Definition 8. A function is **concave up** if its graph is ”bending upwards”.

Definition 9. A function is **concave down** if its graph is ”bending downwards”.

Definition 10. An **inflection point** is a point on the graph where the concavity changes (from up to down, or down to up).

(Precise definitions will come later.)

Concavity and the Rate of Change

Question. When a function is concave up/down, what is happening with its rate of change?

See this Desmos animation (use the a slider to see how the tangent line moves).

Observation: The concavity of a function describes how its rate of change (i.e., the slope of the tangent line) is itself changing.

- **Concave up**: The slope of the tangent line is **increasing**. For example, a function can be increasing at an **increasing rate**.
- **Concave down**: The slope of the tangent line is **decreasing**. For example, a function can be increasing but at a **decreasing rate**.