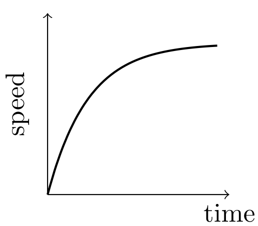
Section 1.4: Limits and End Behavior of Graphs

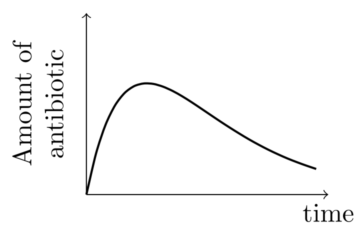
End Behavior of Graphs

Question. Why do we care about a graph's long-term behavior?

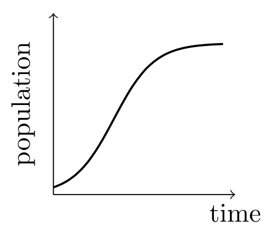
Many real-world phenomena have interesting long-term trends, such as population growth, the concentration of medicine in the bloodstream, or the speed of a falling object.



Skydiving Speed



Antibiotics in Blood



Population Growth

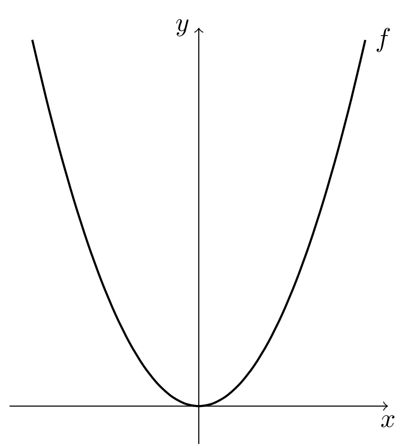
Definition 1. The limit of $f(x)$ at infinity and negative infinity is described as the **end behavior** of the graph of f.

Notation:

- The notation $x \to \infty$ means x gets infinitely large
- The notation $x \to -\infty$ means x gets infinitely large in the negative direction.

Example.

Consider the function $f(x) = x^2$.



As x grows or shrinks, $f(x)$ grows. We write this using **limit notation**:

$$\lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to-\infty} f(x) = \infty$$

Question. How can we calculate limits without a graph?

We can think about what happens when we plug in a very large number for x. Some straightforward limits for n a strictly positive integer:

$$\lim_{x\to\infty} x^n = \infty$$

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & n \text{ even} \\ -\infty, & n \text{ odd} \end{cases}$$

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

$$\lim_{x\to\infty} \sqrt[n]{x} = \infty$$

$$\lim_{x\to-\infty} \sqrt[n]{x} = \begin{cases} \text{undefined}, & n \text{ even} \\ -\infty, & n \text{ odd} \end{cases}$$

Theorem. The end behavior of a polynomial is the same as that of its leading term (term with the highest degree).

We can use this theorem and the equalities above to solve more difficult problems.

Example.

Let $f(x) = x^4 - 3x^2 + 4x - 7$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$$\begin{aligned}\lim_{x\to\infty} x^4 - 3x^2 + 4x - 7 &= \lim_{x\to\infty} x^4 \left(1 - \frac{3}{x^2} + \frac{4}{x^3} - \frac{7}{x^4}\right) \\ &= \lim_{x\to\infty} x^4 \\ &= \infty\end{aligned}$$

Infinite Limits of Rational Functions

Let $f(x)$ be a polynomial with leading term ax^n and $g(x)$ be a polynomial with leading term bx^n. For the rational function

$$R(x) = \frac{f(x)}{g(x)},$$

1. If the degree of f and g are equal, then

$$\lim_{x\to\pm\infty} R(x) = \frac{a}{b}.$$

2. If the degree of g is larger than the degree of f, then

$$\lim_{x\to\pm\infty} R(x) = 0$$

3. If the degree of f is larger than the degree of g, then

$$\lim_{x\to\pm\infty} R(x) = \pm\infty$$