Section 10.3: Graphical Solutions of Change Equations

Equilibrium Solutions

Definition 1 (Equilibrium Solution)**.** An **equilibrium solution** occurs when the rate of change of a function y is zero. This means that the value of y is constant. In other words,

$$\frac{dy}{dx} = 0$$

Example. Consider the change equation $\frac{dy}{dx} = 3y - 6$. Show that the constant function $y = 2$ is an equilibrium solution.

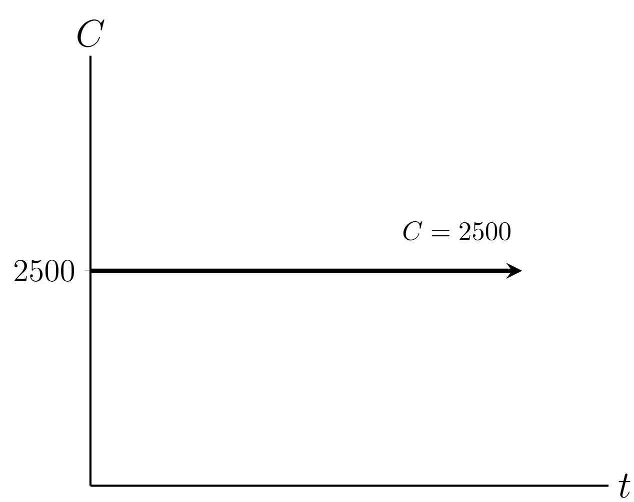
$$\frac{dy}{dx} = 3(2) - 6 = 0$$

Example. Let C denote the number of catfish in a commercial pond at time t measured in months. The catfish reproduce at a rate proportional to the number present, with constant of proportionality $r = 0.2$ per month. Each month 500 catfish are harvested. These facts lead to the change equation

$$\frac{dC}{dt} = .2C - 500.$$

1. Find the equilibrium solution and sketch it.

$$.2C - 500 = 0 \longrightarrow C = 2500.$$



2. Explain the meaning of the solution in terms of the number of catfish and rate of change of the number of catfish.

   If you start with 2500 catfish, the population will stay constant at 2500 catfish indefinitely.

Example. 1. A population Q of ducks in a protected area reproduces at a rate proportional to the current population, with constant of proportionality $r = 0.08$ per year. Limited hunting is allowed. If H ducks per year are allowed to be taken by hunters, then the duck population is governed by the change equation

$$\frac{dQ}{dt} = .08Q - H.$$

Here t is time measured in years. How many ducks per year should be taken by hunters if the duck population is to remain at the level of 800?

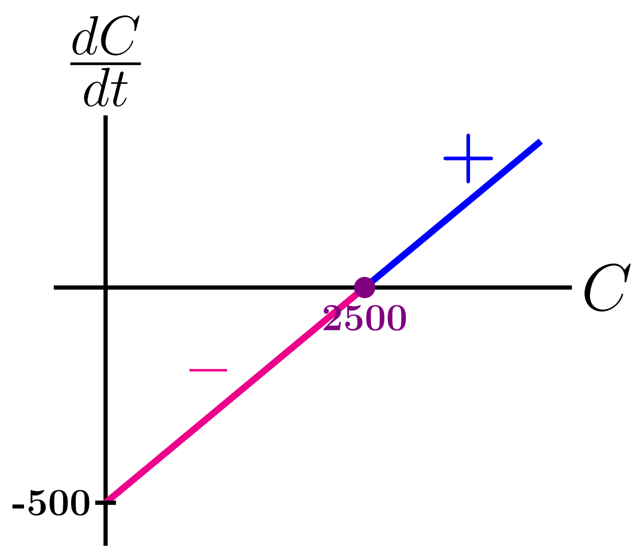
$$0 = .08(800) - H \longrightarrow 0 = 64 - H.$$

So, 64 ducks/year should be taken to keep the population constant.

Graphing Solutions to Rate of Change Equations

Example. In the previous example with the catfish, we had the change equation $\frac{dC}{dt} = .2C - 500$ with equilibrium solution $C = 2500$.

1. Plot the graph of $\frac{dC}{dt}$ versus C. Notice that the change equation is a linear function, with $m = .2$ as the slope. The horizontal intercept



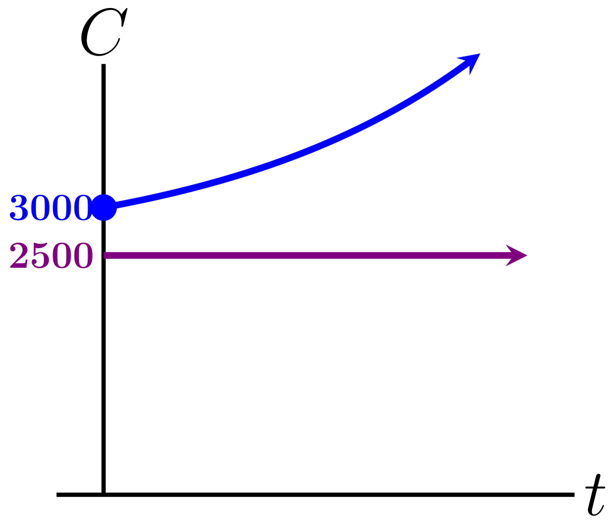
2. Interpret the features of the graph of $\frac{dC}{dt}$ versus C in each of the following regions: $C < 2500, C = 2500,$ and $C > 2500$.

$$C < 2500 : \frac{dC}{dt} < 0, \quad C = 2500 : \frac{dC}{dt} = 0, \quad C > 2500 : \frac{dC}{dt} > 0$$

3. Interpret the features of the graph of C versus t in each of the following regions: $C < 2500, C = 2500,$ and $C > 2500$.

$$C < 2500 : \text{ C is decreasing}, \quad C = 2500 : \text{ C is constant}, \quad C > 2500 : \text{ C is increasing}$$

4. Sketch the graph of the equilibrium solution. Add to it the graph of the solution C versus t in the case where the initial value is $C(0) = 3000$ catfish.



Strategy for Graphing Solutions to Rate of Change Equations

To sketch the solution of a change equation $\frac{dy}{dx} = g(y)$,

1. Find and plot the equilibrium solutions.
2. Plot the graph of $\frac{dy}{dx}$ versus y.
3. Determine the regions where $\frac{dy}{dx}$ is positive and where it is negative. This information shows the regions of increase and regions of decrease for y as a function of x.
4. Given an initial value, use the information from step 3 to sketch the graph of y versus x.

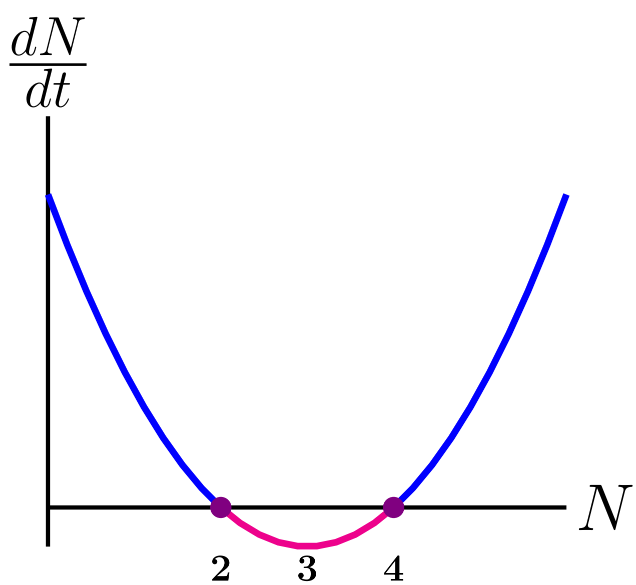
Example. For the change equation $\frac{dN}{dt} = (N-2)(N-4)$,

1. Find the equilibrium solutions.

$$N = 2, N = 4$$

2. Plot the graph of $\frac{dN}{dt}$ versus N.

   Notice that the equation is a quadratic.



At $N = 3$, the slope of $N(t)$ is the steepest in the negative direction. If $N(0) = 3$, the graph is decreasing.

3. Sketch the graph of the equilibrium solutions. Also, make sketches the solution N versus t in the case where the initial value is $N(0) = 2.5$, $N(0) = 5$, $N(0) = -1$, and $N(0) = 3.5$.