Section 2.1: Function Operations

Arithmetic Operations on Functions

We can create new functions by applying standard arithmetic operations. The domain of the new function is the intersection of the domains of the original functions, with one special rule for division.

Addition and Multiplication

The notation for these operations can be written in two ways:

- Addition: $h(x) = f(x) + g(x)$ or $h(x) = (f + g)(x)$
- Multiplication: $h(x) = f(x) \cdot g(x)$ or $h(x) = (fg)(x)$

Domain: The domain of the new function is the intersection of the domains of $f(x)$ and $g(x)$.

Example. Let $f(x) = x^2$ and $g(x) = 2x + 1$.

- $f(x) + g(x) = x^2 + 2x + 1$
- $f(x)g(x) = x^2(2x + 1) = 2x^3 + x^2$
- A constant multiple: $3f(x) = 3x^2$

Division

Division is written as $h(x) = \frac{f(x)}{g(x)}$ or $h(x) = \left(\frac{f}{g}\right)(x)$.
Domain: The domain is the intersection of the domains of $f(x)$ and $g(x)$, **excluding** any x-values for which $g(x) = 0$.

Example. Let $f(x) = x^2$ and $g(x) = 2x + 1$.

- $h(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}$
- The domains of f and g are both $(-\infty, \infty)$. We must exclude the value where $g(x) = 0$, which is $x = -1/2$.
- The domain of $h(x)$ is $(-\infty, -1/2) \cup (-1/2, \infty)$.

Function Composition

Definition 1. Let f and g be functions. The **composition** of f with g, denoted $f \circ g$, is the function defined by $(f \circ g)(x) = f(g(x))$. To calculate it, you first evaluate $g(x)$ and then plug that result into $f(x)$.

Domain of a Composition: The domain of $(f \circ g)(x)$ consists of all x in the domain of g such that the output, $g(x)$, is in the domain of f.

Example. Let $f(x) = x^2 + x$ and $g(x) = 4x$.

- $(f \circ g)(x) = f(g(x)) = (g(x))^2 + g(x) = (4x)^2 + 4x = 16x^2 + 4x$.
- The domain of $g(x)$ is all real numbers.
- The domain of $f(x)$ is all real numbers.
- Therefore, the domain of $(f \circ g)(x)$ is all real numbers $(-\infty, \infty)$.

Example. Let $f(x) = \frac{1}{x}$ and $g(x) = x - 1$.

- $(f \circ g)(x) = f(g(x)) = \frac{1}{g(x)} = \frac{1}{x-1}$.
- The domain of $g(x)$ is all real numbers.
- The domain of $f(x)$ is all real numbers except $x = 0$.
- We must find which x-values make $g(x) = 0$. This occurs at $x - 1 = 0$, so $x = 1$.
- Therefore, the domain of $(f \circ g)(x)$ is all real numbers except $x = 1$.

Decomposition

Decomposition is the reverse of composition: given a function $h(x)$, we find two simpler functions $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

Example. Decompose $h(x) = \sqrt{x^2 + 1}$.

- We can identify an "inner" function, $g(x)$, and an "outer" function, $f(x)$.
- Let the inner function be $g(x) = x^2 + 1$.
- Let the outer function be $f(x) = \sqrt{x}$.
- Then $(f \circ g)(x) = f(g(x)) = \sqrt{x^2 + 1} = h(x)$.

Even and Odd Functions

(Note: This section is not in the textbook)

Definition 2. A function f is

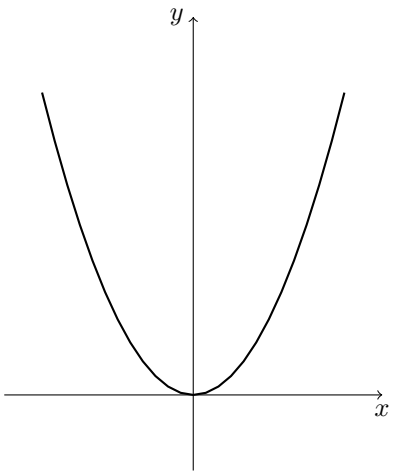
- **Even** if $f(-x) = f(x)$ for all x in its domain.
- **Odd** if $f(-x) = -f(x)$ for all x in its domain.

Example.

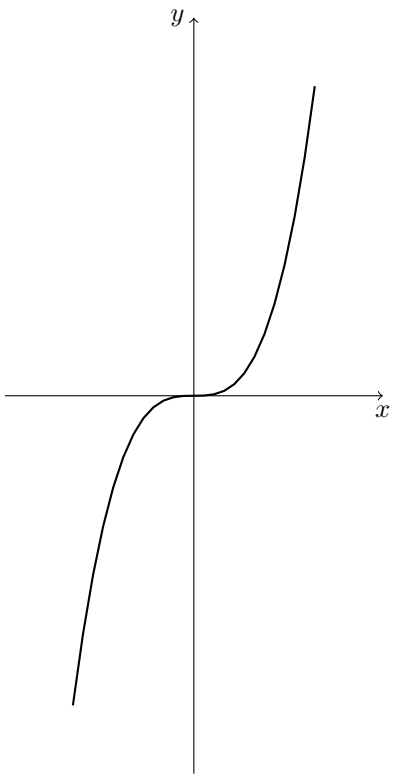
- $f(x) = x^2$ is **even** because $f(-x) = (-x)^2 = x^2 = f(x)$.
- $g(x) = x^3$ is **odd** because $g(-x) = (-x)^3 = -x^3 = -g(x)$.

Graphical Interpretation

- **Even functions** are symmetric with respect to the **y-axis**.
- **Odd functions** are symmetric with respect to the **origin**.



Even Function: $f(x) = x^2$



Odd Function: $g(x) = x^3$

Note: Most functions are neither even nor odd.