Section 2.2: Inverse Functions

The Idea of an Inverse Function

An inverse function is a function that **"undoes"** the action of another function.

Example. If a function tells you your *height* given your *age*, the inverse function would tell you your *age* given your *height*.

Definition 1. A function g is the **inverse function** of f if:

- $g(f(x)) = x$ for all x in the domain of f, **and**
- $f(g(y)) = y$ for all y in the domain of g.

This can also be stated as: $g(y) = x$ if and only if $f(x) = y$.

Notation: The inverse of f is written as f^{-1}.

Example (Verifying Inverses)**.** Let $f(x) = x + 1$ and $g(x) = x - 1$. Are they inverses?

- $g(f(x)) = f(x) - 1 = (x + 1) - 1 = x$.
- $f(g(x)) = g(x) + 1 = (x - 1) + 1 = x$.

Since both compositions equal x, g is the inverse of f (so $g = f^{-1}$).

Properties of Inverses

Domain and Range

The domain and range of a function and its inverse are swapped.

- The **domain** of f^{-1} is the **range** of f.
- The **range** of f^{-1} is the **domain** of f.

Example. Given some values for a function $f(x)$:

x	$f(x)$
-1	2
0	3
7	5
9	-2

To find $f^{-1}(-2)$, we ask "what input to f gives an output of -2?" From the table, we see $f(9) = -2$. Therefore, $f^{-1}(-2) = 9$.

Example. If the domain of f is $[-1, 1]$ and the range of f is $[2, 3]$, then the domain of f^{-1} is $[2, 3]$ and the range of f^{-1} is $[-1, 1]$.

Finding an Inverse Algebraically

Let $y = f(x)$ be a function. To find $g = f^{-1}$, we solve for x in terms of y.

Example. Find the inverse of $f(x) = 3x - 1$.

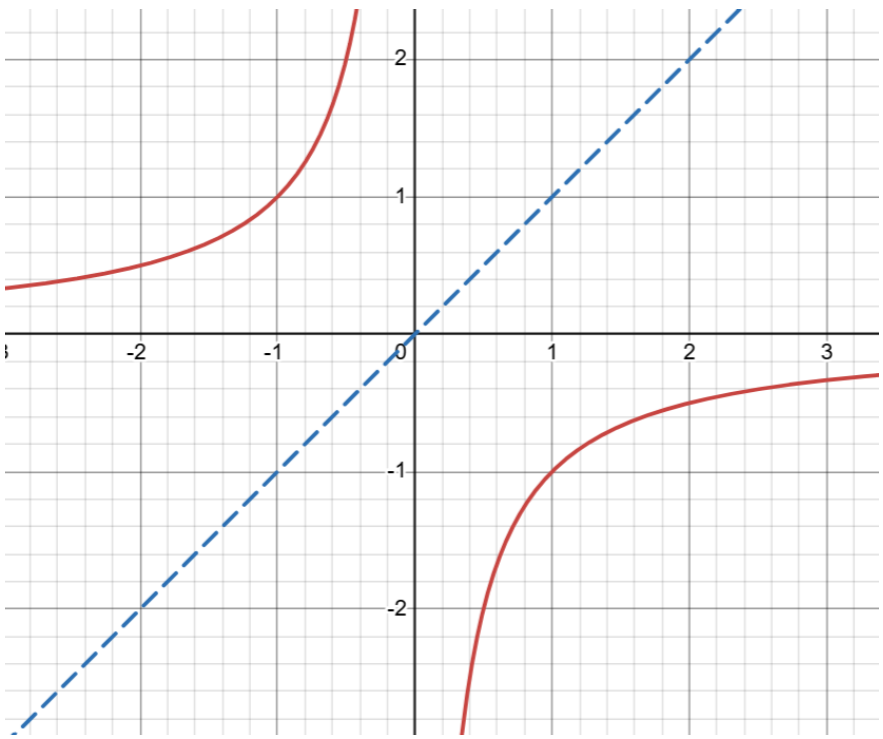
1. Set $y = 3x - 1$.
2. Solve for x:

$$\begin{aligned} y + 1 &= 3x \\ \frac{y+1}{3} &= x \end{aligned}$$

3. The resulting expression is the inverse function: $f^{-1}(y) = \frac{y+1}{3}$.
4. Since the variable name doesn't matter, we can write it as $f^{-1}(x) = \frac{x+1}{3}$.
5. Alternative, we can write $x(y) = \frac{y+1}{3}$.

Graphical Property

The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ across the line $y = x$.

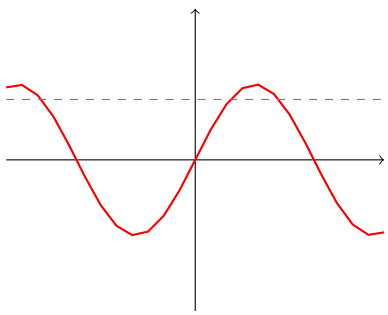


Existence of an Inverse

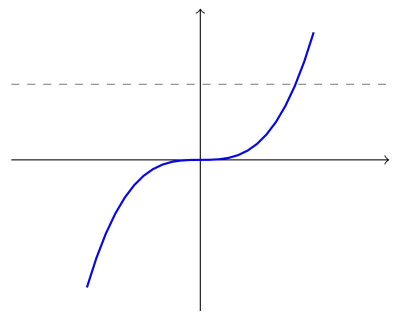
Note that not all functions have an inverse. For a function to have an inverse, each output value must correspond to exactly one input value.

Definition 2. Horizontal Line Test: A function has an inverse if and only if every horizontal line intersects its graph at most once.

Definition 3. A function that passes the horizontal line test is called a **one-to-one** function. Only one-to-one functions have inverses.



No inverse exists (fails HLT).



Inverse exists (passes HLT).

Restricting the Domain

Sometimes a function that is not one-to-one can be made invertible by restricting its domain.

Example. Consider $f(x) = x^2$.

- On its full domain $(-\infty, \infty)$, $f(x) = x^2$ fails the horizontal line test and is not one-to-one.
- However, if we **restrict the domain** to $[0, \infty)$, the function becomes one-to-one.

Let's find the inverse of $h(x) = x^2$ on the domain $[0, \infty)$.

1. Set $y = x^2$.
2. Solve for x: $\sqrt{y} = x$. (We choose the positive root because the domain is $x \geq 0$.)
3. So, $h^{-1}(x) = \sqrt{x}$.