Section 4.1: Logarithms

Motivation and Definition

How do we solve an equation like $10 = 2^t$ for the exponent t? We can't use roots (like we would for $10 = t^2$), so we need a new tool: the inverse of an exponential function.

Definition 1. For $b > 0, b \neq 1, x > 0$, the base-b **logarithm**, written $y = \log_b x$, is the inverse function of the exponential function $f(x) = b^x$.

In other words, the logarithmic statement $y = \log_b x$ is equivalent to the exponential statement $x = b^y$.

Example.

- $\log_{10} 100 = 2$ because $10^2 = 100$.
- $\log_5 \frac{1}{25} = -2$ because $5^{-2} = \frac{1}{25}$.
- $\log_b b = 1$ because $b^1 = b$.

Definition 2.

- **Common Logarithm:** The base-10 logarithm, written as $\log x$.
- **Natural Logarithm:** The base-e logarithm, written as $\ln x$. So, $\ln x = \log_e x$.

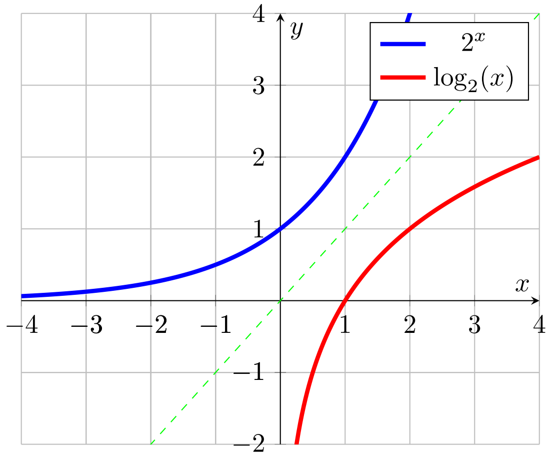
Because logarithms and exponentials are inverses, they "cancel" each other out.

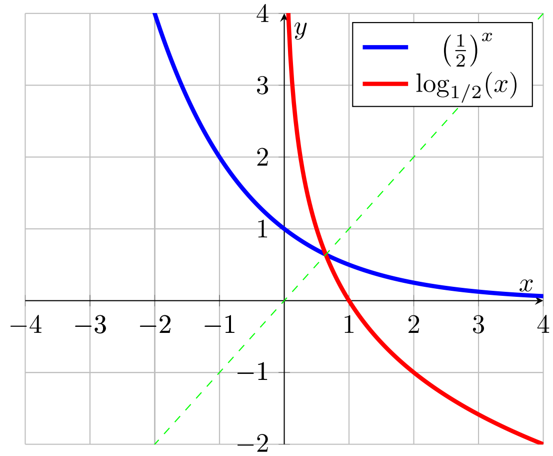
Example.

- $\ln(e) = 1$
- $\log(10^t) = t$
- $b^{\log_b x} = x, x > 0$
- $\log_b(b^x) = x$

Graphs and End Behavior

The graph of a logarithm is the reflection of its corresponding exponential function across the line $y = x$.





Properties of the Logarithm

1. The domain of $\log_b x$ is $(0, \infty)$.
2. The range of $\log_b x$ is $(-\infty, \infty)$.
3. For every base b, the point $(1, 0)$ is on the graph of $\log_b x$ because
$$b^0 = 1 \longleftrightarrow \log_b(1) = 0.$$
4. If $b > 1$, then $\log_b x$ is increasing; if $0 < b < 1$, then $\log_b x$ is decreasing.

Translating between exponentials and logarithms

Example.

- $4^3 = 64$ can be rewritten as $3 = \log_4(64)$.
- $\log(100) = 2$ can be rewritten as $100 = 10^2$

Example. Evaluate $3^{\log_3 5 + \log_3 7}$.

$$\begin{aligned} 3^{\log_3 5 + \log_3 7} &= 3^{\log_3 5} \cdot 3^{\log_3 7} \quad \text{(using exponent rules)} \\ &= 5 \cdot 7 = \mathbf{35} \quad \text{(using inverse property)} \end{aligned}$$

Domain and Change of Base

Domain of Logarithmic Functions

The input (or "argument") of any logarithm must be strictly positive.

Example. Find the domain of $f(x) = \log_5(x - 5)$.

- We need the argument to be greater than zero: $x - 5 > 0$.
- Solving for x, we get $x > 5$.
- The domain is $(5, \infty)$.

Change of Base Formula

To convert a logarithm from a base b to a new base a, use the following formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example. To calculate $\log_{10} x$ using natural logarithms:

$$\log x = \frac{\ln x}{\ln 10}$$

Application: Linearizing Exponential Data

Logarithms are extremely useful for analyzing exponential data. Applying a logarithm ”flattens” or ”linearizes” an exponential curve.

Example. A debt of \$40,000 grows at 7% per year. The function is $f(x) = 40000(1.07)^x$. Notice how the value of $\ln(f(x))$ appears to grow linearly in the table below.

Year (x)	Debt f(x)	$\ln(f(x))^1$
0	\$40,000	156.6
1	\$42,800	157.6
2	\$45,796	158.6
10	\$78,686	166.6
15	\$110,361	171.6

Plotting the logarithm of an exponential function against time produces a straight line.