Section 4.2: Laws of Logarithms

Deriving the Laws of Logarithms

The laws of logarithms are derived directly from the laws of exponents:

$$e^{A+B} = e^A e^B \qquad e^{A-B} = \frac{e^A}{e^B} \qquad e^{pA} = (e^A)^p$$

By letting $A = \ln x$ and $B = \ln y$, we can derive the log laws.

- **Product Rule:**

$$e^{\ln x + \ln y} = e^{\ln x} e^{\ln y} = xy$$

  Taking the natural log of the start and end of this equation gives:

$$\ln(xy) = \ln x + \ln y$$

- **Quotient Rule:**

$$e^{\ln x - \ln y} = \frac{e^{\ln x}}{e^{\ln y}} = \frac{x}{y}$$

  Taking the natural log gives:

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

- **Power Rule:**

$$e^{p \ln x} = (e^{\ln x})^p = x^p$$

  Taking the natural log gives:

$$\ln(x^p) = p \ln x$$

Definition 1 (General Laws of Logarithms)**.** For any base $b > 0$ ($b \neq 1$):

1. **Product Rule:** $\log_b(xy) = \log_b x + \log_b y$
2. **Quotient Rule:** $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. **Power Rule:** $\log_b(x^p) = p \log_b x$

Examples

Condensing Logarithmic Expressions

Example. Combine into a single logarithm.

- $\log_6 4 + \log_6 9 = \log_6(4 \cdot 9) = \log_6(36) = 2$
- $\log_2 48 - \log_2 3 = \log_2\left(\frac{48}{3}\right) = \log_2(16) = 4$
- $\log x - \log(x^2+1) = \log\left(\frac{x}{x^2+1}\right)$
- $\log_2(x-3) + \log_2(5x+1) - \log_2(x) = \log_2\left(\frac{(x-3)(5x+1)}{x}\right)$
- $2\ln x - \ln(1+x) + 3\ln(3-17x) = \ln(x^2) - \ln(1+x) + \ln((3-17x)^3) = \ln\left(\frac{x^2(3-17x)^3}{1+x}\right)$

Expanding Logarithmic Expressions

Example. Expand into a sum/difference of simpler logarithms.

- $\ln(x^2) = 2\ln x$
- $\ln\left(\frac{e^3}{(3-2x)^4}\right) = \ln(e^3) - \ln((3-2x)^4) = 3\ln e - 4\ln(3-2x) = 3 - 4\ln(3-2x)$