Section 4.3: Solving Exponential and Logarithmic Equations

Solving Exponential Equations

The primary strategy is to get the variable out of the exponent by taking a logarithm of both sides.

Example. Solve $5^x = 17$.

$$\begin{aligned} \ln(5^x) &= \ln(17) && \text{Take the natural log of both sides} \\ x\ln(5) &= \ln(17) && \text{Use the power rule for logs} \\ x &= \frac{\ln(17)}{\ln(5)} && \text{Solve for x} \end{aligned}$$

Note: Any base logarithm could be used. By the change of base formula, $\frac{\ln(17)}{\ln(5)} = \frac{\log_b(17)}{\log_b(5)}$ *for any base* b.

Example. Solve $6e^{-x} = e^{2x}$.

$$\begin{aligned} 6 &= \frac{e^{2x}}{e^{-x}} && \text{Isolate the constant} \\ 6 &= e^{2x-(-x)} = e^{3x} && \text{Use exponent rules} \\ \ln(6) &= \ln(e^{3x}) && \text{Take the natural log of both sides} \\ \ln(6) &= 3x && \text{Since } \ln(e^k) = k \\ x &= \frac{\ln(6)}{3} && \text{Solve for x} \end{aligned}$$

Example. Solve $2^{3x} = 5 \cdot 4^x$.

Method 1: Combine exponential terms first.

$$\begin{aligned} (2^3)^x &= 5 \cdot 4^x \\ 8^x &= 5 \cdot 4^x \\ \frac{8^x}{4^x} &= 5 \\ \left(\frac{8}{4}\right)^x &= 5 \\ 2^x &= 5 \end{aligned}$$

$$\ln(2^x) = \ln(5) \implies x\ln(2) = \ln(5) \implies x = \frac{\ln(5)}{\ln(2)}$$

Method 2: Take the logarithm immediately.

$$\begin{aligned} \ln(2^{3x}) &= \ln(5 \cdot 4^x) \\ 3x\ln(2) &= \ln(5) + \ln(4^x) \\ 3x\ln(2) &= \ln(5) + x\ln(4) \\ 3x\ln(2) - x\ln(4) &= \ln(5) \\ x(3\ln(2) - \ln(4)) &= \ln(5) \\ x &= \frac{\ln(5)}{3\ln(2) - \ln(4)} \end{aligned}$$

(This is the same answer, since $3\ln 2 - \ln 4 = \ln(2^3) - \ln 4 = \ln(8/4) = \ln 2$)

Solving Logarithmic Equations

The primary strategy is to use log laws to condense terms, then exponentiate both sides to remove the logs. **Always check your answers** to ensure they are in the domain of the original logarithms.

Example. Solve $\log(2x+4) = \log(x+20)$.

$$\begin{aligned} 10^{\log(2x+4)} &= 10^{\log(x+20)} && \text{Exponentiate both sides with base 10} \\ 2x+4 &= x+20 && \text{Use the inverse property } 10^{\log k} = k \\ x &= 16 && \text{Solve for x} \end{aligned}$$

Check: Both arguments are positive for $x = 16$, so the solution is valid.

Example. Solve $\ln x + \ln(x+2) = \ln 3$.

$$\begin{aligned} \ln(x(x+2)) &= \ln 3 && \text{Condense using the product rule} \\ e^{\ln(x(x+2))} &= e^{\ln 3} && \text{Exponentiate both sides with base e} \\ x(x+2) &= 3 \\ x^2+2x-3 &= 0 && \text{Solve the quadratic equation} \\ (x+3)(x-1) &= 0 \end{aligned}$$

The potential solutions are $x = -3$ and $x = 1$.
Check: The term $\ln x$ in the original equation requires $x > 0$. Therefore, $x = -3$ is not a correct solution. The only valid solution is $x = 1$.

Example. Solve $2 = \ln(1+3^x)$.

$$\begin{aligned} e^2 &= e^{\ln(1+3^x)} && \text{Exponentiate both sides with base e} \\ e^2 &= 1+3^x \\ e^2-1 &= 3^x && \text{Now solve the exponential equation} \\ \ln(e^2-1) &= \ln(3^x) && \text{Take the natural log of both sides} \\ \ln(e^2-1) &= x\ln(3) \\ x &= \frac{\ln(e^2-1)}{\ln(3)} \end{aligned}$$

Application: Doubling Time

Example. An investment grows according to the formula $B(t) = 800(1.07)^t$, where t is in years. How long does it take for the investment to double?
"Doubling" means the balance becomes $2 \times 800 = 1600$.

$$\begin{aligned} 1600 &= 800(1.07)^t \\ 2 &= (1.07)^t && \text{Divide by 800} \\ \ln(2) &= \ln(1.07^t) && \text{Take the natural log of both sides} \\ \ln(2) &= t\ln(1.07) \\ t &= \frac{\ln(2)}{\ln(1.07)} \approx 10.24 \text{ years} \end{aligned}$$

Note: The time it takes to double again (from \$1600 to \$3200) will be the same. This is a key property of exponential growth.