Section 5.1: Quadratic Functions

Polynomials and Quadratics

Definition 1. A **polynomial** of degree n is a function of the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, \ldots, a_0$ are constants and $a_n \neq 0$.

- The **leading term** of f is $a_n x^n$.
- The **leading coefficient** of f is a_n.
- A **zero** is a number c such that $f(c) = 0$.

Definition 2. A **quadratic function** is a polynomial of degree 2, with the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. To find its zeros, we use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant and Complex Zeros

Recall (Complex Numbers)**.**

$$i = \sqrt{-1}, \quad i^2 = -1$$

A complex number is of the form $z = a + bi$, where a, b are real numbers. The **complex conjugate** of z is $\bar{z} = a - bi$

Definition 3. The **discriminant** is the part of the quadratic formula under the square root:

$$d = b^2 - 4ac.$$

It tells us about the nature of the zeros:

- If $d > 0$, there are two distinct real zeros.
- If $d = 0$, there is one real zero.
- If $d < 0$, there are two complex zeros.

Example. Find the discriminant of $f(x) = x^2 - \sqrt{8}x + 2$.

$$d = b^2 - 4ac = (-\sqrt{8})^2 - 4(1)(2) = 0.$$

Note. Complex zeros of polynomials with real coefficients always occur in **conjugate pairs** of the form $a + bi$ and $a - bi$.

Example. Find the zeros of $f(x) = 2x^2 - 4x + 4$.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(4)}}{2(2)} = \frac{4 \pm \sqrt{16 - 32}}{4} \\ &= \frac{4 \pm \sqrt{-16}}{4} = \frac{4 \pm 4i}{4} = 1 \pm i \end{aligned}$$

The zeros are the complex conjugate pair $1 + i$ and $1 - i$.

Factoring and Graphing Quadratics

Theorem 1 (Factor Theorem)**.** For a quadratic function $f(x)$ with leading coefficient a:

- If p and q are distinct zeros, then $f(x) = a(x - p)(x - q)$.
- If p is the only zero, then $f(x) = a(x - p)^2$.

Example. Factor $f(x) = 2x^2 - 4x + 4$. From before and by the Factor Theorem,

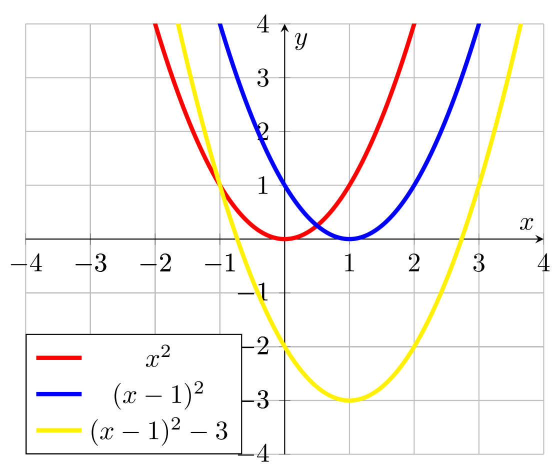
$$f(x) = 2(x - (1 - i))(x - (1 + i)).$$

Graphing

When graphing quadratics, always compare to $y = x^2$.

Example. Sketch $y = (x - 1)^2 - 3$. Note that

$$y = \underbrace{(x-1)^2}_{\text{Shifts right by 1 unit}} \overbrace{-3}^{\text{Shifts down by 3 units}}$$



Definition 4 (Vertex Form)**.** A quadratic function is in **vertex form** if it is written as

$$f(x) = a(x-h)^2 + k.$$

- The vertex of the parabola is at the point (h, k).
- For standard form $f(x) = ax^2 + bx + c$, the vertex occurs at $x = -\frac{b}{2a}$.
- The vertex is a minimum if $a > 0$ and a maximum if $a < 0$.

Application: Projectile Motion

Example. A projectile is launched upwards. After t seconds, its height h is modeled as

$$h(t) = -16t^2 + 64t + 6.$$

1. What is the initial height of the projectile?

   The initial height is when $t = 0$. In other words, $h(0) = 6$ ft

2. At what time does the projectile hit the ground?

   We need to solve $h(t) = 0$. By the quadratic formula,

$$\begin{aligned} t &= \frac{-64 \pm \sqrt{(64)^2 - 4(-16)(6)}}{2(-16)} \\ &= \frac{-64 \pm \sqrt{4480}}{-32} \\ &= \frac{-64 \pm 67}{-32} \\ &\approx -.09, 4.09 \end{aligned}$$

   Notice that $t = -.09$ is outside of the domain. So, it takes approximately 4.1 seconds for the projectile to hit the ground.

3. What is the maximum height that will be reached? When does this occur?

   We know that the vertex is at $t = -\frac{b}{2a} = -\frac{64}{2(-16)} = 2$ and

$$h(2) = -16(2)^2 + 64(2) + 6 = 70.$$

   In other words, the projectile reaches a maximum height of 70 feet at 2 seconds.

4. What is a reasonable domain and range?

   Time cannot be negative, nor does it make sense to include time after the projectile hits the ground. So, a reasonable domain is $[0, 4.1]$. For the range, the height of the projectile cannot be negative, and it does not go higher than 70 feet. So, a reasonable range is $[0, 70]$.

5. Sketch $h(t)$.