Section 5.5: Rational Functions

Definition

Definition 1. A **rational function** is a function of the form $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with no shared real zeros.

- A value c where the denominator is zero, $Q(c) = 0$, is called a **pole** of R.
- A value c where the denominator is zero, $P(c) = 0$, is called a **zero** of R.
- The domain of $R(x)$ is all real numbers except for its poles.

Example. Find the poles, zeros, and domain. of

$$R(x) = \frac{x+1}{x^3 - 2x^2 + x}.$$

Zeros: Find x such that

$$P(x) = x + 1 = 0.$$

In other words, the only zero of R is $x = -1$.

Poles: Find x such that

$$Q(x) = x^3 - 2x^2 + x = 0.$$

Factoring, we find this is equivalent to solving

$$x(x-1)^2 = 0.$$

In other words, the poles of R are $x = 0$ and $x = 1$.

Domain: The domain of R is all real numbers except the poles. That is,

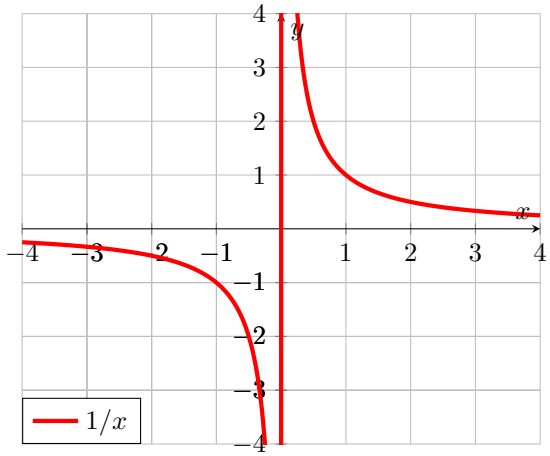
$$(-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

Asymptotes

Definition 2. The line $x = a$ is a **vertical asymptote** (VA) of $f(x)$ if, as x approaches a from the left or right (or both), $f(x)$ approaches positive or negative infinity.

Note that vertical asymptotes only occur at poles.

Example. $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$.



Recall. A **horizontal asymptote** (HA) of a function $R(x)$ is a line $y = L$ such that

$$R(x) \to L \text{ as } x \to \infty \text{ or } x \to -\infty.$$

A common method is to compare the leading terms of the numerator and denominator:

$$\lim_{x\to\pm\infty} R(x) = \lim_{x\to\pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$$

Example. Find the horizontal asymptotes of $f(x) = \frac{x+3}{2x^2-4}$.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x+3}{2x^2-4} = \lim_{x\to\pm\infty} \frac{x}{2x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

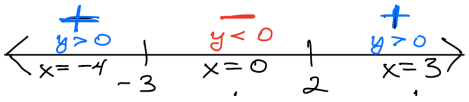
Note. A graph can cross its horizontal asymptote, but not its vertical asymptote.

Sign Charts

A rational function can only change its sign at its zeros or vertical asymptotes. We can use a **sign chart** to determine where the function is positive or negative.

Example. Make a sign chart for the function $f(x) = \frac{x+3}{2x-4}$.

Note that the domain is when $2x - 4 \neq 0$, i.e. $x \neq 2$. In other words, there is a vertical asymptote at $x = 2$. Additionally, there are zeros when $x + 3 = 0$, or when $x = -3$. So, we check some test points on each side of the VAs and zeros.

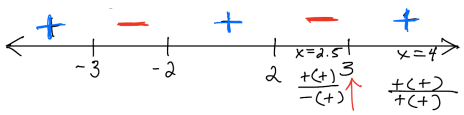


We see that f is positive when $x < -3$, negative when $-3 < x < 2$, and positive again when $x > 2$.

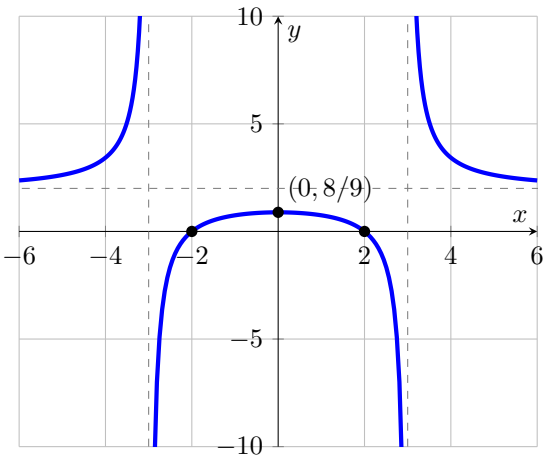
Complete Graphing Examples

Example. Sketch the graph of $f(x) = \frac{2x^2-8}{x^2-9}$.

1. **Poles / Vertical Asymptotes:** $x^2 - 9 = 0 \implies (x-3)(x+3) = 0$. The VAs are at $x = 3$ and $x = -3$.
2. **Zeros:** $2x^2 - 8 = 0 \implies x^2 = 4 \implies x = \pm 2$.
3. **Sign Chart:**

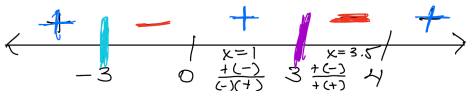


4. **Horizontal Asymptote:** $\lim_{x\to\infty} \frac{2x^2}{x^2} = 2$. The HA is at $y = 2$.
5. **Y-intercept:** $f(0) = \frac{-8}{-9} = \frac{8}{9}$.
6. **Sketch:**



Example. Sketch the graph of $g(x) = \frac{2x^2-8x}{x^2-9}$.

1. **Poles / VAs:** $x^2 - 9 = 0 \implies x = \pm 3$.
2. **Zeros:** $2x^2 - 8x = 0 \implies 2x(x-4) = 0 \implies x = 0, 4$.
3. **Sign Chart:**



4. **HA:** $\lim_{x\to\infty} \frac{2x^2}{x^2} = 2$. HA is at $y = 2$.
5. **Y-intercept:** $g(0) = 0$.
6. **Sketch:**