Section 8.1: Trigonometric Identities

Elementary Identities

Definition 1 (Recall: Reciprocal and Quotient Identities)**.**
Reciprocal Identities:

$$\sec(x) = \frac{1}{\cos(x)} \qquad \cos(x) = \frac{1}{\sec(x)}$$
$$\csc(x) = \frac{1}{\sin(x)} \qquad \sin(x) = \frac{1}{\csc(x)}$$
$$\cot(x) = \frac{1}{\tan(x)} \qquad \tan(x) = \frac{1}{\cot(x)}$$

Quotient Identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1$$

Example. Verify that $\tan(t) = \frac{1}{\cot(t)}$.

$$\frac{1}{\cot(t)} = \frac{1}{\cos(t)/\sin(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t)$$

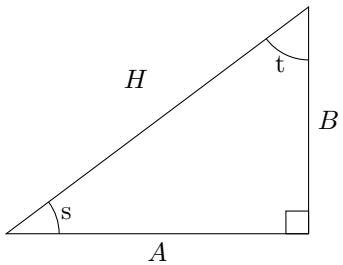
Example. Express $\tan(\theta)$ in terms of $\sin(\theta)$ if $0 \leq \theta \leq \frac{\pi}{2}$.

By Pythagorean Identity, $\cos^2(\theta) = 1 - \sin^2(\theta)$. Since θ is in the first quadrant, we take the positive root. So,

$$\tan(\theta) = \frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}}.$$

Cofunction Identities

Note.



From a right triangle with angles t, s, and $\frac{\pi}{2}$, we know $s + t = \frac{\pi}{2}$, so $s = \frac{\pi}{2} - t$. Moreover

- $\sin(s) = \frac{B}{H}$ and $\cos(s) = \frac{A}{H}$
- $\sin(t) = \frac{A}{H}$ and $\cos(t) = \frac{B}{H}$

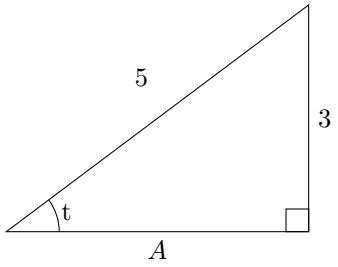
In other words, $\sin(t) = \cos(s) = \cos(\pi/2 - t)$ and $\cos(t) = \sin(s) = \sin(\pi - t)$.

More generally, we have

Definition 2 (Cofunction Identities)**.**

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) \qquad \cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$
$$\tan(x) = \cot\left(\frac{\pi}{2} - x\right) \qquad \cot(x) = \tan\left(\frac{\pi}{2} - x\right)$$
$$\sec(x) = \csc\left(\frac{\pi}{2} - x\right) \qquad \csc(x) = \sec\left(\frac{\pi}{2} - x\right)$$

Example. Given $\sin(t) = \frac{3}{5}$ with t in the first quadrant, find all six trig functions for $\frac{\pi}{2} - t$.



First, find the values for t. $A^2 + 3^2 = 5^2 \implies A^2 = 16 \implies A = 4$. So, $\tan(t) = \frac{3}{4}$.

Using the cofunction identities:

$$\cos\left(\frac{\pi}{2} - t\right) = \sin(t) = \frac{3}{5}$$
$$\cot\left(\frac{\pi}{2} - t\right) = \tan(t) = \frac{3}{4}$$

The other values can be found using a similar approach

Even/Odd Identities

Recall (Even and Odd Functions)**.**

- **Even functions:** $f(-x) = f(x)$ (symmetric about y-axis)
- **Odd functions:** $f(-x) = -f(x)$ (symmetric about origin)

Definition 3 (Even/Odd Identities)**.**

- **Even:** Cosine and Secant are even.

$$\cos(-x) = \cos(x) \qquad \sec(-x) = \sec(x)$$

- **Odd:** All other trig functions are odd.

$$\sin(-x) = -\sin(x) \qquad \csc(-x) = -\csc(x)$$

$$\tan(-x) = -\tan(x) \qquad \cot(-x) = -\cot(x)$$

Example. Is $g(x) = \sec(x)\tan(x)$ even or odd?

$$\begin{aligned} g(-x) &= \sec(-x)\tan(-x) \\ &= (\sec(x))(-\tan(x)) \\ &= -\sec(x)\tan(x) \\ &= -g(x) \end{aligned}$$

Therefore, $g(x)$ is **odd**.

Verifying Identities

To prove the identity $A = B$, we can either show $A = B$ or $B = A$, or we can show that $A = C$ and $B = C$, and so $A = B$. We cannot just "solve the equation"! We can only do this if we know the equation is true already.

Example. Show that

$$2\sec^2(t) = \frac{1}{1+\sin(t)} + \frac{1}{1-\sin(t)}.$$

We will work with the Right Hand Side (RHS):

$$\begin{aligned} \text{RHS} &= \frac{1}{1+\sin(t)} + \frac{1}{1-\sin(t)} \\ &= \frac{(1-\sin(t))}{(1+\sin(t))(1-\sin(t))} + \frac{(1+\sin(t))}{(1-\sin(t))(1+\sin(t))} \\ &= \frac{1-\sin(t)+1+\sin(t)}{1-\sin^2(t)} \\ &= \frac{2}{1-\sin^2(t)} \\ &= \frac{2}{\cos^2(t)} && \text{by Pythagorean Identity: } \cos^2(t) = 1-\sin^2(t) \\ &= 2\left(\frac{1}{\cos(t)}\right)^2 && \text{by Reciprocal Identity: } \sec(t) = \frac{1}{\cos(t)} \\ &= 2\sec^2(t) = \text{LHS} \end{aligned}$$

Example. Show that

$$\frac{\sin(t)}{1+\cos(t)} = \frac{1-\cos(t)}{\sin(t)}.$$

$$\begin{aligned}
\text{LHS} &= \frac{\sin(t)}{1+\cos(t)} \cdot \frac{1-\cos(t)}{1-\cos(t)} \\
&= \frac{\sin(t)(1-\cos(t))}{1-\cos^2(t)} \\
&= \frac{\sin(t)(1-\cos(t))}{\sin^2(t)} && \text{by Pythagorean Identity: } \cos^2(t) = 1-\sin^2(t) \\
&= \frac{1-\cos(t)}{\sin(t)} \\
&= \text{RHS}
\end{aligned}$$