Section 8.2: Sum and Difference Formulas

Definition 1 (Sine and Cosine)**.**

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

(Note the sign flip for cosine)

Example. Calculate $\sin\left(\frac{7\pi}{12}\right)$ in exact form.

First, break the angle into a sum of known angles:

$$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

Now, use the sum formula for sine:

$$\begin{aligned}
\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

Definition 2 (Tangent and Cotangent)**.**

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\alpha \pm \cot\beta}$$

Example. Calculate $\tan\left(\frac{7\pi}{12}\right)$ in exact form.

First, break the angle into a sum of known angles:

$$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

Recalling that $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, we can use the sum formula for tangent:

$$\begin{aligned}
\tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) &= \frac{\tan(\pi/4) + \tan(\pi/3)}{1 - \tan(\pi/4)\tan(\pi/3)} \\
&= \frac{1+\sqrt{3}}{1-\sqrt{3}}
\end{aligned}$$

Example. Verify $\tan(\alpha+\beta) = \frac{1+\tan(\beta)\cot(\alpha)}{\cot(\alpha)-\tan(\beta)}$.

$$\begin{aligned}\text{RHS} &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\cdot\frac{1/\tan\alpha}{1/\tan\alpha}\\ &= \frac{\frac{\tan\alpha}{\tan\alpha}+\frac{\tan\beta}{\tan\alpha}}{\frac{1}{\tan\alpha}-\frac{\tan\alpha\tan\beta}{\tan\alpha}}\\ &= \frac{1+\cot\alpha\tan\beta}{\cot\alpha-\tan\beta}\end{aligned}$$

What about angles not in the first quadrant?

Example. Let $\sin(t_1) = -\frac{1}{3}$ with $\pi < t_1 < \frac{3\pi}{2}$ and $\cos(t_2) = \frac{4}{5}$ with $\frac{3\pi}{2} < t_2 < 2\pi$. Find $\sin(t_1+t_2)$.

1. Find $\cos(t_1)$ and $\sin(t_2)$:

- For t_1 (Quad III): $A^2+(-1)^2 = 3^2 \implies A^2 = 8 \implies A = \pm 2\sqrt{2}$. In QIII, $A = -2\sqrt{2}$. So, $\cos(t_1) = \frac{-2\sqrt{2}}{3}$.
- For t_2 (Quad IV): $4^2+B^2 = 5^2 \implies B^2 = 9 \implies B = \pm 3$. In QIV, $B = -3$. So, $\sin(t_2) = \frac{-3}{5}$.

2. Use the sum formula:

$$\begin{aligned}\sin(t_1+t_2) &= \sin(t_1)\cos(t_2)+\cos(t_1)\sin(t_2)\\ &= \left(-\frac{1}{3}\right)\left(\frac{4}{5}\right)+\left(\frac{-2\sqrt{2}}{3}\right)\left(\frac{-3}{5}\right)\\ &= \frac{-4}{15}+\frac{6\sqrt{2}}{15}\\ &= \frac{-4+6\sqrt{2}}{15}\end{aligned}$$

As an exercise, find $\cos(t_1+t_2)$.

Definition 3 (Product-to-Sum Formulas)**.**

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]\\ \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]\end{aligned}$$

Example. Find $\sin(52.5^\circ)\cos(7.5^\circ)$.

$$\begin{aligned}\sin(52.5^\circ)\cos(7.5^\circ) &= \frac{1}{2}[\sin(52.5^\circ+7.5^\circ)+\sin(52.5^\circ-7.5^\circ)]\\ &= \frac{1}{2}[\sin(60^\circ)+\sin(45^\circ)]\\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{3}+\sqrt{2}}{4}\end{aligned}$$

Definition 4 (Sum-to-Product Formulas)**.**

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Example. Find $\sin(75°) + \sin(15°)$.

$$\begin{aligned}\sin(75°) + \sin(15°) &= 2\sin\left(\frac{75° + 15°}{2}\right)\cos\left(\frac{75° - 15°}{2}\right)\\ &= 2\sin(45°)\cos(30°)\\ &= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}\end{aligned}$$