Section 8.4: Inverse Trig Functions

Arcsine

Definition 1. The **arcsine** function, written as $\arcsin(x)$ or $\sin^{-1}(x)$, is the inverse of the sine function restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

In other words:

$$\arcsin(x) = \theta \quad \text{if and only if} \quad \sin(\theta) = x \quad \text{AND} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Note. This restricted domain is necessary for the function to pass the horizontal line test, which is required for an inverse function to exist.

Example. Determine the exact value of the following:

1. $\sin^{-1}\left(\frac{1}{2}\right) = \theta$

   We are looking for an angle θ such that $\sin(\theta) = \frac{1}{2}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

   So, the only possibility is $\theta = \frac{\pi}{6}$.

2. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$

   We are looking for θ such that $\sin(\theta) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

   So, the only possibility is $\theta = -\frac{\pi}{4}$.

Example (Function Composition)**.** Determine the exact value of the following:

1. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

2. $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

3. $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

   First, evaluate the inside: $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

   Now find $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$. We need $\sin(\theta) = -\frac{\sqrt{3}}{2}$ where θ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   $\theta = -\frac{\pi}{3}$.

   So, $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$.

   Note.

- $\sin(\arcsin(x)) = x$ for all x.
- $\arcsin(\sin(x)) = x$ only if x is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Arccosine

Definition 2. The **arccosine** function, written as $\arccos(x)$ or $\cos^{-1}(x)$, is the inverse of the cosine function restricted to the domain $[0, \pi]$.

In other words:

$$\arccos(x) = \theta \quad \text{if and only if} \quad \cos(\theta) = x \quad \text{AND} \quad 0 \leq \theta \leq \pi$$

Example. Determine the exact value of the following:

1. $\cos^{-1}\left(\frac{1}{2}\right) = \theta$ We are looking for an angle θ such that $\cos(\theta) = \frac{1}{2}$ and $0 \leq \theta \leq \pi$.

   So, the only possibility is $\theta = \frac{\pi}{3}$

2. $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \theta$

   We are looking for an angle θ such that $\cos(\theta) = -\frac{\sqrt{2}}{2}$ and $0 \leq \theta \leq \pi$.

   So, the only possibility is $\theta = \frac{3\pi}{4}$

Example (Function Composition)**.** Determine the exact value of the following:

1. $\cos\left(\arccos\left(\frac{1}{2}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

2. $\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

3. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

   First, evaluate the inside: $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

   Now find $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$. We need $\cos(\theta) = -\frac{\sqrt{3}}{2}$ where θ is in $[0, \pi]$.

   $\theta = \frac{5\pi}{6}$.

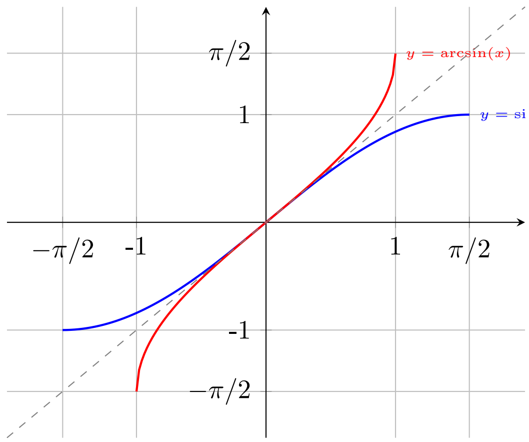
Note.

- $\cos(\arccos(x)) = x$ for all x in $[-1, 1]$.
- $\arccos(\cos(x)) = x$ only if x is in $[0, \pi]$.

Graphing Inverse Functions

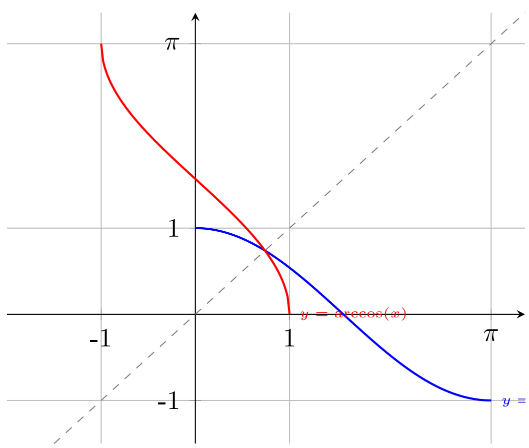
Example (Graphing Arcsine)**.** $y = \arcsin(x)$ is the reflection of $y = \sin(x)$ (on its restricted domain) across the line $y = x$.

- **Domain:** $[-1, 1]$
- **Range:** $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Example (Graphing Arccosine)**.** $y = \arccos(x)$ is the reflection of $y = \cos(x)$ (on its restricted domain) across the line $y = x$.

- **Domain:** $[-1, 1]$
- **Range:** $[0, \pi]$



Arctangent

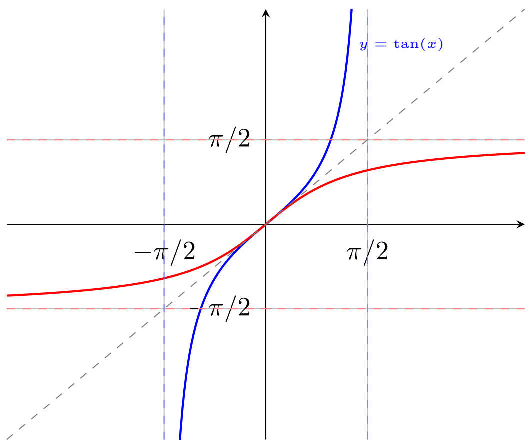
Definition 3. The **arctangent** function, written as $\arctan(x)$ or $\tan^{-1}(x)$, is the inverse of the tangent function restricted to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

In other words:

$$\arctan(x) = \theta \quad \text{if and only if} \quad \tan(\theta) = x \quad \text{AND} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Example (Graphing Arctangent)**.** $y = \arctan(x)$ is the reflection of $y = \tan(x)$ (on its restricted domain) across the line $y = x$.

- **Domain:** $(-\infty, \infty)$
- **Range:** $(-\frac{\pi}{2}, \frac{\pi}{2})$



More Examples

Example (Other Inverses)**.** Determine the exact value of the following:

1. $\arctan(-\sqrt{3}) = \theta$
   We need $\tan(\theta) = -\sqrt{3}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
   $\theta = -\frac{\pi}{3}$
2. $\text{arccsc}(-2) = \theta$
   This is equivalent to $\csc(\theta) = -2$, which means $\sin(\theta) = -\frac{1}{2}$.
   $\theta = -\frac{\pi}{6}$

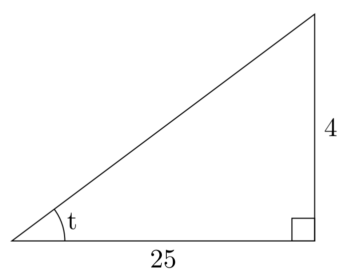
Example (More Function Compositions)**.**

1. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$
   First, let $\theta = \arccos(-\frac{1}{2})$. Then, $\cos(\theta) = -\frac{1}{2}$ with θ in $[0, \pi]$. This means $\theta = \frac{2\pi}{3}$.
   Now, find $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
   $\theta = \frac{2\pi}{3}$
2. $\text{arccot}\left(\tan\left(-\frac{\pi}{3}\right)\right)$
   First, $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.
   Now, let $\theta = \text{arccot}(-\sqrt{3})$. This means $\cot(\theta) = -\sqrt{3}$, or $\tan(\theta) = -\frac{1}{\sqrt{3}}$.
   $\theta = \frac{5\pi}{6}$.

Example (Trig Equations)**.** Find the solutions on the interval $[0, 2\pi]$.

1. $\sin(x) = -\frac{1}{\sqrt{3}}\cos(x)$
   $\frac{\sin(x)}{\cos(x)} = -\frac{1}{\sqrt{3}} \implies \tan(x) = -\frac{1}{\sqrt{3}}$
   The angles in $[0, 2\pi]$ where $\tan(x) = -\frac{1}{\sqrt{3}}$ are $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.
2. $\arctan(x) = \frac{\pi}{4}$
   By definition, $\tan\left(\frac{\pi}{4}\right) = x$. So, $x = 1$.

Example (Application)**.** The rim of a standard basketball goal is 10 feet high. A point guard's eyes are 6 feet above the floor. She stands 25 feet from a point directly below the front of the rim. At what angle t (in degrees) must she incline her eyes to look directly at the front of the rim?



The height from her eyes to the rim is $10 - 6 = 4$ feet. The horizontal distance is 25 feet. This forms a right triangle.

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{25}$$

To find the angle t, we use the arctangent function:

$$t = \arctan\left(\frac{4}{25}\right)$$