Section 8.5: Solving Trigonometric Equations

Basic Trigonometric Equations

Example (Example 1)**.**

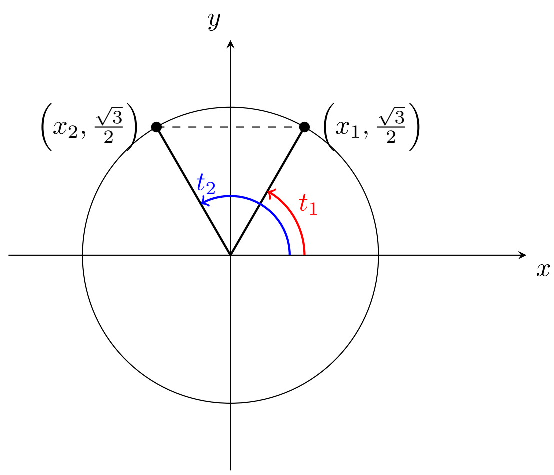
Consider the equation $2\sin(t) = \sqrt{3}$.

1. **Find the solutions on** $[0, 2\pi)$**.**

   First, isolate the trigonometric function:

   $$\sin(t) = \frac{\sqrt{3}}{2}$$

   We look for angles where the sine is $+\frac{\sqrt{3}}{2}$.

   This occurs in Quadrant I and Quadrant II. For angle t_2, note that the reference angle is $\frac{\pi}{3}$.

   Solutions: $t_1 = \frac{\pi}{3}$ and $t_2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

2. **Find all solutions.**

   Since the period of sine is 2π, we add multiples of 2π to our specific solutions.

   $$t = \frac{\pi}{3} + 2\pi k \quad \text{and} \quad t = \frac{2\pi}{3} + 2\pi k$$

   where k is any integer (positive or negative). Recall that adding a multiple of 2π gives coterminal angles.

Example (Example 2)**.**

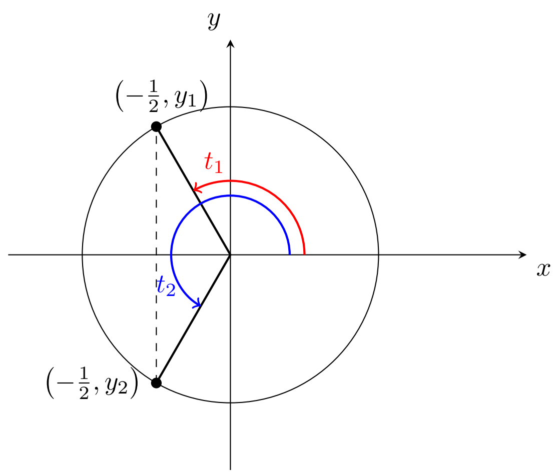
Consider the equation $4\cos(t) = -2$.

1. **Find the solutions on** $[0, 2\pi)$**.**

   First, isolate the trigonometric function:

   $$\cos(t) = -\frac{2}{4} = -\frac{1}{2}$$

   We look for angles where the cosine value is $-\frac{1}{2}$.

   This occurs in Quadrant II and Quadrant III. Note that the reference angle is $\frac{\pi}{3}$.

   Solutions: $t_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and $t_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

2. **Find the solutions on** $[-\pi, \pi)$**.**

   $\frac{2\pi}{3}$ is in the interval. However, $\frac{4\pi}{3}$ is not, so we subtract 2π:

   $$\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$$

   Solutions: $t = \frac{2\pi}{3}, -\frac{2\pi}{3}$.

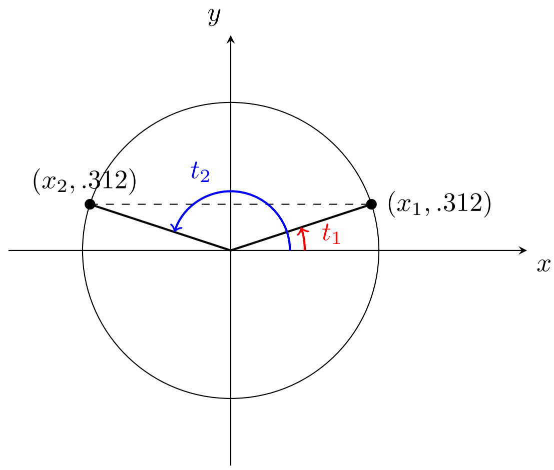
3. **Find all solutions.**

   Since the period of cosine is 2π, we add multiples of 2π to our specific solutions.

   $$t_1 = \frac{2\pi}{3} + 2\pi k \quad \text{and} \quad t_2 = \frac{4\pi}{3} + 2\pi k$$

Example (Example 3)**.**

Find all solutions of the equation $\sin(t) = 0.312$ on $[0, 2\pi)$.



Since 0.312 does not correspond to a special angle, we represent the solutions using arcsine.
Solutions: $t_1 = \arcsin(.312)$ and $t_2 = \pi - \arcsin(.312)$.

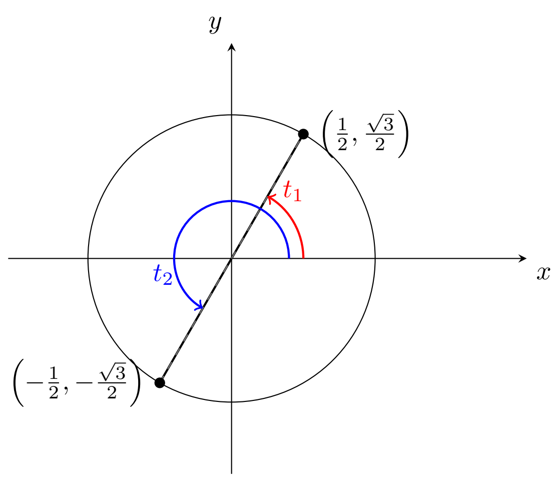
Example (Example 4)**.**

Find all solutions of the equation $\tan(t) = \sqrt{3}$.

1. **Find the solutions on** $[0, 2\pi)$**.**

   We know that $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Since the result $\sqrt{3}$ is positive, sine and cosine must have the same sign.

   We look for angles in Quadrant I (both positive) and Quadrant III (both negative) where the quotient of the coordinates is $\sqrt{3}$.



   Solutions: $t_1 = \frac{\pi}{3}$ and $t_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

2. **Find all solutions.**

   Notice that these solutions are exactly π apart. Since the period of tangent is π, we do not need two separate general equations. We can combine them into one.

$$t = \frac{\pi}{3} + \pi k$$

   where k is any integer.

Multiple Angles

When the argument of the trig function is a multiple of t (like $2t$ or $3t$), some additional work is required.

Example (Example 5)**.**

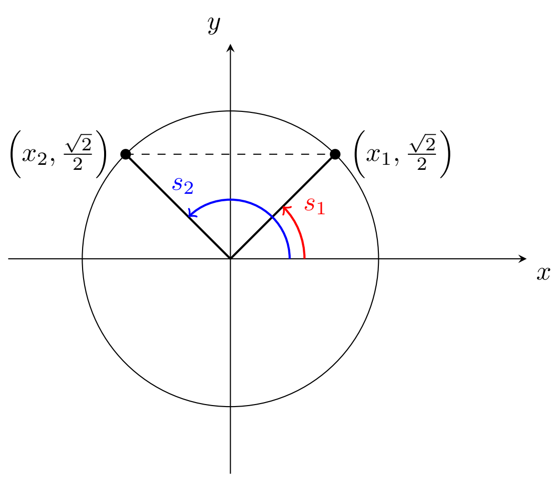
Find all solutions of $\sin(2t) = \frac{\sqrt{2}}{2}$.

1. **Find the solutions for $s = 2t$ on $[0, 2\pi)$.**

   First, let $s = 2t$ and isolate the trigonometric function:

$$\sin(s) = \frac{\sqrt{2}}{2}$$

   We look for angles s where sine is $+\frac{\sqrt{2}}{2}$. This occurs in Quadrant I and Quadrant II.



   Solutions for s: $s_1 = \frac{\pi}{4}$ and $s_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

2. **Find the general solutions for t.**

   We use the general solutions for s and substitute back $s = 2t$. Since the period of sine is 2π, we add $2\pi k$ to each solution:

$$s_1 = 2t_1 = \frac{\pi}{4} + 2\pi k \quad \text{and} \quad s_2 = 2t_2 = \frac{3\pi}{4} + 2\pi k$$

   Dividing both sides of both equations by 2, we have

$$t_1 = \frac{\pi}{8} + \pi k \quad \text{and} \quad t_2 = \frac{3\pi}{8} + \pi k$$

3. **Find the solutions on $[0, 2\pi)$.**

   We substitute integer values for k until the angle exceeds 2π.

   - For $t_1 = \frac{\pi}{8} + \pi k$:
     - $k = 0 : t = \frac{\pi}{8}$
     - $k = 1 : t = \frac{\pi}{8} + \pi = \frac{\pi}{8} + \frac{8\pi}{8} = \frac{9\pi}{8}$
   - For $t_2 = \frac{3\pi}{8} + \pi k$:
     - $k = 0 : t = \frac{3\pi}{8}$
     - $k = 1 : t = \frac{3\pi}{8} + \pi = \frac{3\pi}{8} + \frac{8\pi}{8} = \frac{11\pi}{8}$

   Solutions on $[0, 2\pi)$: $t = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$.

Example (Example 6).

$$\text{Find all solutions of } \cot(3t) = \sqrt{3}.$$

1. **Find the solutions for $s = 3t$ on $[0, 2\pi)$.**

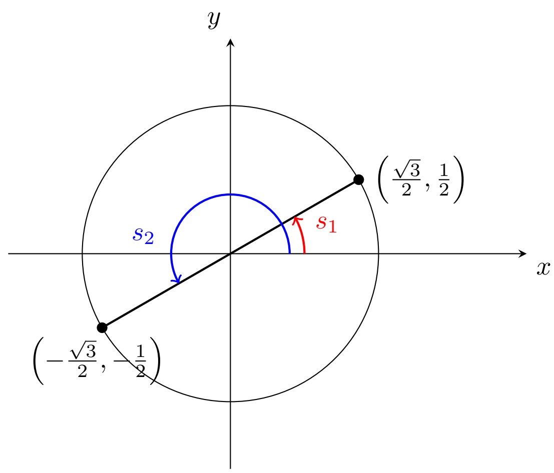
   First, rewrite in terms of tangent:

   $$\tan(3t) = \frac{1}{\cot(3t)} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

   Let $s = 3t$. We isolate the trigonometric function:

   $$\tan(s) = \frac{\sqrt{3}}{3}$$

   We look for angles s where the tangent is positive. This occurs in Quadrant I and Quadrant III.

   Solutions for s: $s_1 = \frac{\pi}{6}$ and $s_2 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

2. **Find the general solutions for t.**

   We use the general solutions for s and substitute back $s = 3t$. Notice s_1 and s_2 are exactly π apart. Since the period of tangent is π, we do not need two separate general equations. We can combine them into one:

   $$s = 3t = \frac{\pi}{6} + \pi k$$

   Dividing both sides by 3, we have:

   $$t = \frac{\pi}{18} + \frac{\pi k}{3} = \frac{\pi + 6\pi k}{18}$$

3. **Find the solutions on $[0, 2\pi)$.**

   We substitute integer values for k until the angle exceeds 2π.

   - $k = 0 : t = \frac{\pi}{18}$
   - $k = 1 : t = \frac{7\pi}{18}$
   - $k = 2 : t = \frac{13\pi}{18}$
   - $k = 3 : t = \frac{19\pi}{18}$
   - $k = 4 : t = \frac{25\pi}{18}$
   - $k = 5 : t = \frac{31\pi}{18}$

   Solutions on $[0, 2\pi)$: $t = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$.

Solving More Complex Equations

Example (Example 7)**.**

Solve the equation $4\sin^2(x) = 3$.

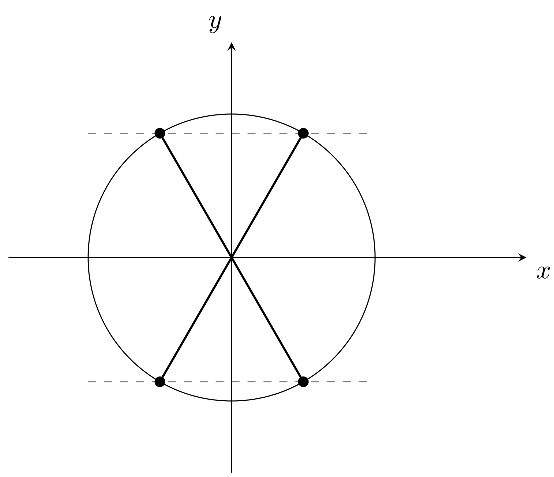
1. **Find the solutions on** $[0, 2\pi)$**.**

   First, isolate the squared term and take the square root:

   $$\sin^2(x) = \frac{3}{4} \implies \sin(x) = \pm\frac{\sqrt{3}}{2}$$

   We need to consider two cases:

   - Case 1: $\sin(x) = \frac{\sqrt{3}}{2}$ (Positive y-values: Quadrant I and II)
   - Case 2: $\sin(x) = -\frac{\sqrt{3}}{2}$ (Negative y-values: Quadrant III and IV)

   - Positive Case: $x_1 = \frac{\pi}{3}$ and $x_2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
   - Negative Case: $x_3 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and $x_4 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

   Solutions on $[0, 2\pi)$: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

2. **Find all solutions.**

   We can write the general solution for each specific solution by adding $2\pi k$. However, notice the symmetry:

   - $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ are exactly π apart.
   - $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$ are exactly π apart.

   We can condense the four solutions into two by using multiples of π:

   $$x = \frac{\pi}{3} + \pi k \quad \text{and} \quad x = \frac{2\pi}{3} + \pi k$$

Example (Example 8).

Solve the equation $\sin(2x) = 0.3\cos(x)$. Find all solutions.

1. **Use identities and factor.**

   First, use the double angle identity $\sin(2x) = 2\sin(x)\cos(x)$:

   $$2\sin(x)\cos(x) = 0.3\cos(x)$$

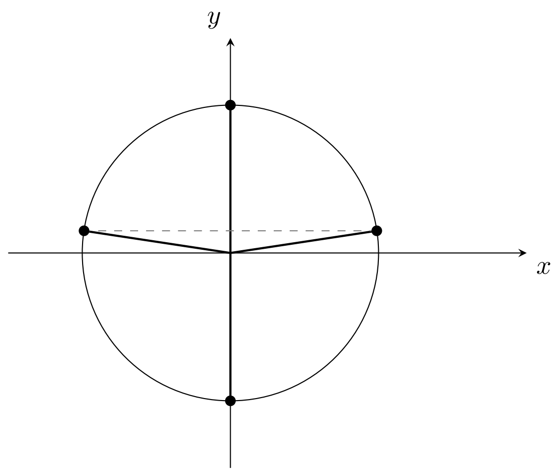
   Set to zero and factor (do not divide by cosine!)

   $$2\sin(x)\cos(x) - 0.3\cos(x) = 0$$

   $$\cos(x)(2\sin(x) - 0.3) = 0$$

   This gives us two separate equations to solve:

   - Case 1: $\cos(x) = 0$
   - Case 2: $2\sin(x) - 0.3 = 0 \implies \sin(x) = 0.15$



2. **Solve Case 1:** $\cos(x) = 0$**.**

   Cosine is zero at the top and bottom of the unit circle.

   $$x = \frac{\pi}{2} + \pi k$$

3. **Solve Case 2:** $\sin(x) = 0.15$**.**

   We look for angles where sine is 0.15. Since 0.15 is positive, solutions are in Quadrant I and Quadrant II.

   Since 0.15 is not a standard unit circle value, we express the solutions using arcsine.

   - Quadrant I: $x = \arcsin(0.15) + 2\pi k$
   - Quadrant II: $x = \pi - \arcsin(0.15) + 2\pi k$

4. **Combine all solutions.**

   The complete set of solutions is:

   $$x = \frac{\pi}{2} + \pi k, \quad x = \arcsin(0.15) + 2\pi k, \quad x = \pi - \arcsin(0.15) + 2\pi k$$

Example (Example 9)**.**

Solve the equation $2\sin^2(x) + \sin(x) = 1$. Find all solutions.

1. **Use identities and factor.**

   First, subtract 1 from both sides to set the equation to zero:

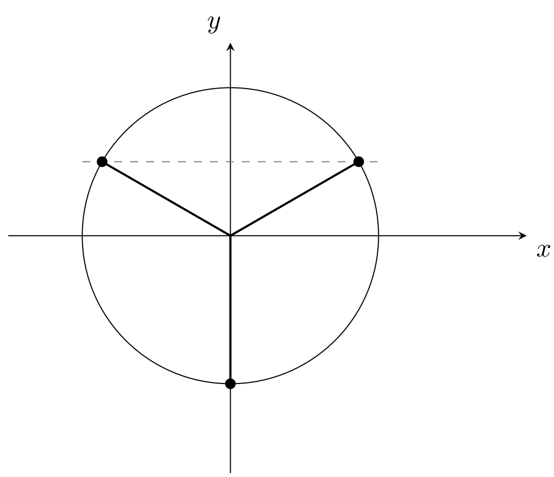
   $$2\sin^2(x) + \sin(x) - 1 = 0$$

   This is a quadratic equation in sine. Factor it just like $2u^2 + u - 1 = (2u - 1)(u + 1)$:

   $$(2\sin(x) - 1)(\sin(x) + 1) = 0$$

   This gives us two separate equations to solve:

   - Case 1: $2\sin(x) - 1 = 0 \implies \sin(x) = \frac{1}{2}$
   - Case 2: $\sin(x) + 1 = 0 \implies \sin(x) = -1$



2. **Solve Case 1:** $\sin(x) = \frac{1}{2}$**.**

   We look for angles where sine is $\frac{1}{2}$. The reference angle is $\frac{\pi}{6}$.

   - Quadrant I: $x = \frac{\pi}{6} + 2\pi k$
   - Quadrant II: $x = \pi - \frac{\pi}{6} + 2\pi k = \frac{5\pi}{6} + 2\pi k$

3. **Solve Case 2:** $\sin(x) = -1$**.**

   Sine is -1 at the bottom of the unit circle.

   - y-axis: $x = \frac{3\pi}{2} + 2\pi k$

4. **Combine all solutions.**

   The complete set of solutions is:

   $$x = \frac{\pi}{6} + 2\pi k, \quad x = \frac{5\pi}{6} + 2\pi k, \quad x = \frac{3\pi}{2} + 2\pi k$$

Example (Example 10)**.**

Solve the equation $\cos(2x)[\sqrt{3}\tan(2x) - 1] = 0$. Find all solutions.

1. **Set each factor to zero.**

   We separate the equation into two parts:

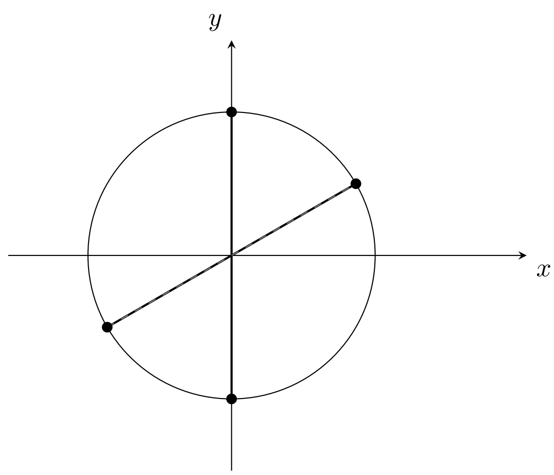
   $$\cos(2x) = 0 \quad \text{or} \quad \sqrt{3}\tan(2x) - 1 = 0$$

   Isolate the trigonometric function in the second equation:

   $$\sqrt{3}\tan(2x) = 1 \implies \tan(2x) = \frac{1}{\sqrt{3}}$$

   Let $s = 2x$. This gives us two separate equations to solve:

   - Case 1: $\cos(s) = 0$. The x-coordinate on the unit circle is 0 (points on the y-axis).
   - Case 2: $\tan(s) = \frac{1}{\sqrt{3}}$. The slope is positive (Quadrant I and III). Reference angle is $\frac{\pi}{6}$.



2. **Solve Case 1:** $\cos(2x) = 0$**.**

   The angles where cosine is zero are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

   - $s = 2x = \frac{\pi}{2} + 2\pi k \implies x = \frac{\pi}{4} + \pi k$
   - $s = 2x = \frac{3\pi}{2} + 2\pi k \implies x = \frac{3\pi}{4} + \pi k$

   (Note: These can be combined into a single expression $x = \frac{\pi}{4} + \frac{\pi k}{2}$).

3. **Solve Case 2:** $\tan(2x) = \frac{1}{\sqrt{3}}$**.**

   The angles are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.

   - $s = 2x = \frac{\pi}{6} + 2\pi k \implies x = \frac{\pi}{12} + \pi k$
   - $s = 2x = \frac{7\pi}{6} + 2\pi k \implies x = \frac{7\pi}{12} + \pi k$

   (Note: These can also be combined into $x = \frac{\pi}{12} + \frac{\pi k}{2}$).

4. **Combine all solutions.**

   The complete set of solutions is:

   $$x = \frac{\pi}{4} + \pi k, \quad x = \frac{3\pi}{4} + \pi k, \quad x = \frac{\pi}{12} + \pi k, \quad x = \frac{7\pi}{12} + \pi k$$